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SYLLABUS

Unit I

Sequences - Bounded sequences - Monotonic Sequences – Convergent Sequences – Divergent and Oscillating Sequences – The Algebra of limits.

Chapter 1: Sections 1.1 - 1.7

Unit II

Behaviour of Monotonic Sequences – Some theorem on limits – Sub sequences – Limit points – Cauchy sequences.

Chapter 2: Sections 2.1 – 2.5

Unit III

Series of positive terms: Infinite series – Comparison test.

Chapter 3: Sections 3.1, 3.2

Unit IV

Kummer's test – Root test – Integral Test.

Chapter 4: Sections 4.1 - 4.3

Unit V

Series of Arbitrary terms: Alternative series – Absolute convergence – Tests for convergence of series of arbitrary terms.

Chapter 5: Sections 5.1 - 5.3

TEXT BOOK

Issac and Dr. Arumugam S, Sequences and Series and Trigonometry (2014), New Gamma Publishing house.

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Unit I

Sequences - Bounded sequences - Monotonic Sequences – Convergent Sequences – Divergent and Oscillating Sequences – The Algebra of limits.

Chapter 1: Sections 1.1 - 1.7

1.Sequences:

1.1 Introduction:

A great deal of analysis is concerned with sequences and series. Consider the following collection of real numbers given by $1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{n}, \ldots$ .... In this collection the first element is 1 , the second element is $\frac{1}{2}$, the third element is $\frac{1}{3}$ and so on. This is an example of a sequence of real numbers. We may think of a sequence as any arrangement of elements where we can say which element is first, which is second, which is third and so on. In other words the elements of a sequence are labelled with the elements of **N** preserving their order. In general such a labelling can be done by means of a function f whose domain is **N**. If the range of f is a subset of an arbitray set X, we get a sequence of elements of X. Throughout this chapter we deal with sequences of real numbers.

1.2. Sequences:

Definition:

Let $f: \mathbf{N} \rightarrow \mathbf{R}$ be a function and let $f(n) = a_n$. Then $a_1, a_2, a_3, \ldots \ldots, a_n,$ ..... is called the sequence in **R** determined by the function f and is denoted by (a_n). a_n is called the n^{th} term of the sequence.

The range of the function f, which is a subset of **R**, is called the range of the sequence.

Examples:

1. The function $f: \mathbf{N} \rightarrow \mathbf{R}$ given by $f(n) = \boldsymbol{n}$ determines the sequence $1,2,3, \quad , n, \ldots$
2. The function $f: \mathbf{N} \rightarrow \mathbf{R}$ given by $f(n) = n^2$ determines the sequence $1,4,9 \ldots\ldots, n^2,$ .....

3. The function $f: \mathbf{N} \to \mathbf{R}$ given by $f(n) = (-1)^n$ determines the sequence $-1, 1, -1, 1, \dots \dots$. Thus the terms of a sequence need not be distinct. The range of this sequence is $\{1, -1\}$. Thus we see that the range of a sequence may be finite or infinite.
4. The sequence $((-1)^{n+1})$ is given by $1, -1, 1, -1 \dots$. The range of this sequence is also $\{1, -1\}$. However we note that the sequence $((-1)^n)$ and $((-1)^{n+1})$ are different. The first sequence starts with -1 and the second sequence starts with 1.
5. The constant function $f: \mathbf{N} \to \mathbf{R}$ given by $f(n) = 1$ determines the sequence 1,1,1, Such a sequence is called a constant sequence.
6. The function $f: \mathbf{N} \to \mathbf{R}$ given by $f(n) = \begin{cases} \frac{1}{2}n \text{ if } n \text{ is even} \\ \frac{1}{2}(1-n) \text{ if } n \text{ is odd} \end{cases}$ determines the sequence $0, 1, -1, 2, -2, \dots \dots, n, -n \dots \dots$. The range of this sequence is $\mathbf{Z}$.
7. The function $f: N \to \mathrm{R}$ given by $f(n) = \frac{n}{n+1}$ determines the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \dots \dots, \frac{n}{n+1}, \dots \dots$
8. The function $f: \mathbf{N} \to \mathbf{R}$ given by $f(n) = \frac{1}{n}$ determines the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots \dots, \frac{1}{n}, \dots \dots$
9. The function $f: \mathbf{N} \to \mathbf{R}$ given by $f(n) = 2n + 3$ determines the sequence $5, 7, 9, 11, \dots \dots$
10. Let $x \in \mathbf{R}$. The function $f: \mathbf{N} \to \mathbf{R}$ given by $f(n) = x^{n-1}$ determines the geometric sequence $1, x, x^2, \dots \dots, x^n, \dots \dots$
11. The sequence $(-n)$ is given by $-1, -2, -3, \dots \dots, -n, \dots$. The range of this sequence is the set of all negative integers.
12. A sequence can also be described by specifying the first few terms and stating a rule for determining a_n in terms of the previous terms of the sequence. For example, let $a_1 = 1, a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$.

    Then $a_3 = a_2 + a_1 = 2; a_4 = a_3 + a_2 = 3$ and so on. we thus obtain the sequence $1, 1, 2, 3, 5, 8, 13, \dots$. This sequence is called Fibonacci's sequence.
13. Let $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{(2 + a_n)}$. This defines the sequence $\sqrt{2}, \sqrt{(2+\sqrt{2})}, \dots \dots$

Exercises 1:

1. Write the first five terms of each of the following sequences.

   (a) $\left(\frac{(-1)^n}{n}\right)$ (b) $\left(\frac{2}{3}\left(1-\frac{1}{10^n}\right)\right)$ (c) $\left(\frac{\cos nx}{n^2+x^2}\right)$ (d) $\left(\frac{(-1)^{n+1}}{n!}\right)$

   (e) $\left(\frac{1-(-1)^n}{n^3}\right)$ (f) $\left(\frac{2n^2+1}{2n^2-1}\right)$ (g) $(n!)$ (h) $f(n)=\begin{cases} n & \text{if } n \\ & n \text{ is odd} \\ 1/n & \text{if } n \end{cases}$ is even.

   (i) $a_1 = 1$ and $a_{n+1} = \sqrt{(2+a_n)}$

2. Determine the range of the following sequences.

   (a) (n) (b) $(2n)$ (c) $(2n-1)$ (d) $(1+(-1)^n)$

   (e) The constant sequence $a, a, a,$

   (f) $f(n) = \begin{cases} 1 \text{ if } n \text{ is odd} \\ 1/n \text{ if } n \text{ is even} \end{cases}$

   (g) $f(n) = \left[\frac{n}{4}\right]$ whert : denotes the integral part of x.

1.3. Bounded Sequences:

Definition:

A sequence (a_n) is said to be bounded above if there exist a real number k such that

$a_n \leq k$ for all $n \in$ N. Then k is called an upper bound of the sequence (a_n).

A sequence (a_n) is said to be bounded below if there exists a real number k such that $a_n \geq k$ for all n. Then k is called a lower bound of the sequence (a_n).

A sequence (a_n) is said to be a bounded sequence if it is both bounded above and below.

Note:

A sequence (a_n) is bounded iff there exists a real number $k \geq 0$. that $|a_n| \leq k$ for all n

Examples:

1. Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots \ldots, \frac{1}{n}, \ldots..$ Here 1 is the $l.u.b$ and 0 is the $g.l.b$. It is a bounded sequence.

2. The sequence $1,2,3,\ldots\ldots,n,\ldots\ldots$ is bounded below but not bounded above. 1 is the $g.l.b$ of the sequence.

3. The sequence $-1,-2,-3,\ldots..,-n$ ..... is bounded above but not bounded below. -1 is the $l.u.b$ of the sequence.

4. $1,-1,1,-1,\ldots$ .... . is a bounded sequence. 1 is the $l.$ u.b. and -1 is the $g.l.b$ of the sequence.

5. Any constant sequence is a bounded sequence. Here $l.u.b = g.l.b =$ the constant term of the sequence.

Exercises:

1. Give examples of sequences (a_n) such that

   (a) (a_n) is bounded above but not bounded below.

   (b) (a_n) is bounded below but not bounded above.

   (c) (a_n) is a bounded sequence.

   (d) (a_n) is neither bounded above nor bounded below.

2. Determine the $l.u.b$ and $g.l.b$ of the following sequences if they exist.

   (a) $2,-2,1,-1,1,-1,$

   (b) $1,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{4}},\ldots\ldots,\frac{1}{\sqrt{n}},\ldots..$

   (c) $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots\ldots,\frac{n}{n+1},\ldots..$

   (d) $1,-1,2,-2,3,-3,\ldots\ldots,n,-n,\ldots\ldots$

   (e) $1,\frac{1}{2},3,\frac{1}{4},5,\frac{1}{6},\ldots.,(2n-1),\frac{1}{2n},\ldots..$

   (f) $1,\frac{1}{2},1,\frac{1}{3},1,\frac{1}{4},\ldots\ldots$

   (g) $(1+n+n^2)$

   (b) $(-n^2)$.

1.4. Monotonic Sequences:

Definition:

A sequence (a_n) is said to be monotonic increasing if $a_n \leq a_{n+1}$ for all n. (a_n) is said to be monotonic decreasing if $a_n \geq a_{n+1}$ ' for all n. (a_n) is said to be strictly monotonic increasing if $a_n < a_{n+1}$ for all n and strictly monotonic decreasing if $a_n > a_{n+1}$ for all n. (a_n) is said to be it is either monotonic increasing or monotonic decreasing.

Examples:

1. 1,2,2,3,3,3,4,4,4,4, ... is a monotonic increasing sequence.
2. $1,2,3,4, \ldots\ldots, n,$ is a strictly monotonic increasing sequence.
3. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots \frac{1}{n}, \ldots..$ is a strictly monotonic decreasing sequence.
4. The sequence (a_n) given by $1, -1, 1, -1, 1, \ldots.$ is neither monotonic increasing nor decreasing. Hence (a_n) is not a monotonic sequence.
5. $\left(\frac{2n-7}{3n+2}\right)$ is a monotonic increasing sequence.

   Proof: $a_n - a_{n+1} = \frac{2n-7}{3n+2} - \frac{2(n+1)-7}{3(n+1)+2} = \frac{-25}{(3n+2)(3n+5)} < 0. \therefore a_n < a_{n+1}.$

   Hence the sequence is monotonic increasing.

6. Consider the sequence (a_n) where

$(a_n) = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$. Clearly (a_n) is a monotonic increasing sequence.

Note:

A monotonic increasing sequence (a_n) is bounded below and a_1 is the g.l. b of the sequence. A monotonic decreasing sequence (a_n) is bounded ab ove and a_1 is the $l.u.b$ of the sequence.

Problem 1.

Show that if (a_n) is a monotonic sequence then $\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$ is also a monotonic sequence.

Solution:

Let (a_n) be a monotonic increasing sequence.

Let (a_n) be a

$\therefore a_1 \leq a_2 \leq a_3 \leq \cdots .. \leq a_n \leq \cdots$ .... (1)

$b_n = \frac{a_1+a_2+\cdots+a_n}{n}$

$$b_{n+1} - b_n = \frac{a_1 + \cdots .. + a_{n+1}}{n+1} - \frac{a_1 + \cdots .. + a_n}{n}$$
$$= \frac{na_{n+1} - (a_1 + \cdots ... + a_n)}{n(n+1)}$$
$$\geq \frac{na_{n+1} - (a_n + a_n + \cdots .. + a_n)}{n(n+1)} \text{ (by (1))}$$
$$= \frac{n(a_{n+1} - a_n)}{n(n+1)}$$
$$\geq 0. \text{ (by (1))}$$

$\therefore b_{n+1} \geq b_n.$

$\therefore (b_n)$ is monotoic increasing.

The proof is similar if (a_n) is monotonic decreasing.

Exercises.

1. Give an example of a sequence (a_n) such that (a_n) is
   (a) monotonic increasing and bounded above.
   (b) monotonic increasing and not bounded above.
   (c) monotonic decreasing and bounded below.
   (d) monotonic decreasing and not bounded below.
2. Determine which of the following sequences are monotonic.
   (a) $(\log n)$
   (b) $((-1)^{n+1}n)$
   (c) $\left(2 + \frac{1}{n}\right)$
   (d) $\left(\frac{1}{2^n}\right)$
   (e) $\left(\frac{1}{n!}\right)$

(f) $\left(\frac{(-1)^n}{n}\right)$

(g) $.6, .66, .666$ (h) $2, 1.9, 1.8,$

3. If (a_n) and (b_n) are two monotonic increasing (decreasing) sequences show that $(a_n + b_n)$ is also monotonic increasing (decreasing).

4. . If (a_n) is monotonic increasing show that (λa_n) is increasing if λ is positive and (λa_n) is decreasing if λ is negative.

1.5. Convergent Sequences:

Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots \dots, \frac{1}{n}, \dots$ ….. We observe that as n increases $\frac{1}{n}$ approaches zero. In fact by raking the value of n sufficiently large, we can bring $\frac{1}{n}$ as close to 0 as we want. This is roughly what we mean when we say that the sequence $(1/n)$ converges to 0 or 0 is the limit of this sequence. This idea is formulated mathematically in the following definition.

Definition:

A sequence (a_n) is said to converge to a number l if given $\varepsilon > 0$ there exists a positive integer m such that $|a_n - l| < \varepsilon$ for all $n \geq m$. We say that l is the limit of the sequence and we write $\lim_{n \to -\infty} a_n = l$ or $(a_n) \to l$.

Note. 1.

$(a_n) \to l$ iff given $\varepsilon > 0$ there exists a natural number m such that $a_n \in (l - \varepsilon, l + \varepsilon)$ for all $n \geq m$ (i.e.), All but a finite number of terms of the sequence lie within the interval $(l - \varepsilon, l + \varepsilon)$.

Note. 2

The above definition does not give any method of finding the limit of a sequence. In many cases, by observing the sequence carefully, we can guess whether the limit exists or not and also the value of the limit.

Theorem 1:

A sequence cannot converge to two different limits.

Proof:

Let (a_n) be a convergent sequence.

If possible let l_1 and l_2 be two distinct limits of (a_n).

Let $\varepsilon > 0$ be given. Since $(a_n) \to l_1$, there exists a natural number n_1 such that

$$|a_n - l_1| < \frac{1}{2}\varepsilon \text{ for all } n \geq n_1$$

Since $(a_n) \to l_2$, there exists a natural number n_2 such lame Pri $|a_n - l_2| < \frac{1}{2}\varepsilon$ for all $n \geq n_2$.

Let $m = \max\{n_1, n_2\}$.

Then $|l_1 - l_2| = |l_1 - a_m + a_m - l_2|$

$$\leq |a_m - l_1| + |a_m - l_2|$$
$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon (\text{ by 1 and 2})$$
$$= \varepsilon.$$

$\therefore |l_1 - l_2| < \varepsilon$ and this is true for every $\varepsilon > 0$.

Clearly this is possible if and only if $l_1 - l_2 = 0$. Hence $l_1 = l_2$.

Example 1:

$\lim\limits_{n \to -\infty} \frac{1}{n} = 0$ (or) $\left(\frac{1}{n}\right) \to 0$.

Proof:

Let $\varepsilon > 0$ be given.

Then $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$ if $n > \frac{1}{\varepsilon}$.

Hence if we choose $\boldsymbol{m}$ to be any natural number such that $\boldsymbol{m} > \frac{1}{\varepsilon}$ then $\left|\frac{1}{n} - 0\right| < \varepsilon$ for all $n \geq m$.

$$\therefore \lim_{n \to \infty} \frac{1}{n} = 0$$

Note:

If $\varepsilon = 1/100$, then m can be chosen to be any natural number grater than 100. In this example the choice' of m depends on the given ε and $[1/\varepsilon] + 1$ is the smallest value of m that satisfies the requirements of the definition.

Example 2:

The constant sequence 1,1,1, ……… converges to 1.

Proof:

Let $\varepsilon > 0$ be given.
Let the given sequence be denoted by (a_n).
Then $a_n = 1$ for all n.

$\therefore |a_n - 1| = |1 - 1| = 0 < \varepsilon$ for all $n \in \mathbf{N}$

$\therefore |a_n - 1| < \varepsilon$ for all $n \geq m$ where m can be chosen to be any natural number.

$\therefore \lim_{n\to\infty} a_n = 1$

Note:

In this example, the choice of m does not depend on the given ε.

Example 3:

$\lim_{n\to\infty} \frac{n+1}{n} = 1.$

Proof:

Let $\varepsilon > 0$ be given.
Now, $\left|\frac{n+1}{n} - 1\right| = \left|1 + \frac{1}{n} - 1\right| = \left|\frac{1}{n}\right|.$
$\therefore$ If we choose m to be any natural number greater than $\frac{1}{\varepsilon}$ we have,

$$\left|\frac{n+1}{n}-1\right| < \varepsilon \text{ for all } n \geq m$$

$$\therefore \lim_{n\to\infty} \frac{n+1}{n} = 1$$

Example 4:

$\lim_{n\to-\infty} \frac{1}{2^n} = 0.$

Proof:

Let $\varepsilon > 0$ be given.

Then $\left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} < \frac{1}{n}$ (since $2^n > n$ for all $n \in$ N).

$\therefore \left|\frac{1}{2^n} - 0\right| < \varepsilon$ for all $n \geq m$ where m is any natural number

greater than $1/\varepsilon$

$\therefore \lim_{n\to-\infty} \frac{1}{2^n} = 0$

Example 5:

The sequence $((-1)^n)$ is not convergent.

Proof:

Suppose the sequence $((-1)^n)$ converges to l.

Then, given $\varepsilon > 0$, there exists a natural number m such that $|(-1)^n - l| < \varepsilon$ for all $n \geq m$.

l $|< \varepsilon$ tor all nzm.

$\therefore |(-1)^m - (-1)^{m+1}| = |(-1)^m - l + l - (-1)^{m+1}|$

$\leq |(-1)^m - l| + |(-1)^{m+1} - l|$

$< \varepsilon + \varepsilon = 2\varepsilon$

But $|(-1)^m - (-1)^{m+1}| = 2.$

$\therefore 2 < 2\varepsilon$ i.e., $1 < \varepsilon$ which is a contradiction since $\varepsilon > O$; arbitrary.

$\therefore$ The sequence $((-1)^n)$ is not convergent.

Theorem 2:

Any convergent sequence is a bounded sequence.

Proof:

Let (a_n) be a convergent sequence.

Let $\lim_{n\to\infty} a_n = l$.

Let $\varepsilon > 0$ be given. Then there exists $m \in N$ such thatit $|a_n - l| < \varepsilon$ for all $n \geq m$.

$\therefore |a_n| < |l| + \varepsilon$ for all $n \geq m$

Now, let $k = \max\{|a_1|, |a_2|, \dots \dots, |a_{m-1}|, |l| + \varepsilon)$

Then $|a_n| \leq k$ for all n.

$\therefore (a_n)$ is a bounded sequence.

Note:

The converse of the above theorem is not true. For example, the sequence $((-1)^n)$ is a bounded sequence. However, it is not a convergent sequence.

Exercises:

1. Prove that $\lim_{n\to-\infty} \frac{1}{n^2} = 0$.
2. Prove that $\lim_{n\to-\infty} \left(1 + \frac{1}{n!}\right) = 1$.
3. Prove that $\lim_{n\to-\infty} \frac{2n+1}{2n} = 1$.
4. Prove that the following sequences are not convergent.
   (a) $((-1)^n n)$.
   (b) (n^2).

1.6. Divergent and Oscillating Sequences:

We now proceed to classify sequences which are not convergent as follows.

1. Sequences diverging to ∞
2. Sequences diverging to $-\infty$

3. Finitely oscillating sequences.

4. Infinitely oscillating sequences.

Definition:

A sequence (a_n) is said to diverge to ∞ if given any real number $k > 0$, there exists $m \in$ N such that $a_n > k$ for all $n \geq m$. In symbols we write $(a_n) \to \infty$ or $\lim_{n\to\infty} a_n = \infty$.

Note:

$(a_n) \to \infty$ iff given any real number $k > 0$ there exists $m \in$ N

such that $a_n \in (k, \infty)$ for all $n \geq m$.

Example 1:

$(n) \to \infty$.

Proof.

Let $k > 0$ be any given real number.

Choose $\boldsymbol{m}$ to be any natural number such that $\boldsymbol{m} > \boldsymbol{k}$.

Then $n > k$ for all $n \geq m$.

$\therefore (n) \to \infty$.

Example 2:

$(n^2) \to \infty$.

Proof:

Let $k > 0$ be any given real number.

Choose $\boldsymbol{m}$ to be any natural number such that $\boldsymbol{m} > \sqrt{k}$.

Then $n^2 > k$ for all $n \geq m$.

$\therefore (n^2) \to \infty$

Example 3:

$(2^n) \to \infty$.

Proof:

Let $k > 0$ be any given real number.

Then $2^n > k \Leftrightarrow n\log 2 > \log k$.

$\Leftrightarrow n > (\log k)/\log 2$

Hence if we choose m to be any natural number such that $m > (\log k)/\log 2$, then $2^n > k$ for all $n \geq m$.

$\therefore (2^n) \to \infty$.

Definition:

A sequence (a_n) is said to diverge to $-\infty$ if given any ral wisk $k < 0$ there exists $m \in$ N such that $a_n < k$ for all $n \geq m$. In symbols we write $\lim_{n\to\infty} a_n = -\infty$ or $(a_n) \to -\infty$.

Note:

$(a_n) \to -\infty$ iff given any real number $k < 0$, there exists $m \in N$ such that $a_n \in (-\infty, k)$ for all $n \geq m$.

A sequence (a_n) is said to be divergent if exists

$(a_n) \to \infty$ or $(a_n) \to -\infty$

Theorem 3:

$(a_n) \to \infty$ iff $(-a_n) \to -\infty$.

proof:

Let $(a_n) \to \infty$.

Let $k < 0$ be any given real number. Since $(a_n) \to \infty$ there exists $m \in$ N such that $a_n > -k$ for all $n \geq m$.

$\therefore -a_n < k$ for all $n \geq m$

$\therefore (-a_n) \to -\infty$

Similarly we can prove that if $(-a_n) \to -\infty$ then $(a_n) \to \infty$.

Examples.

The sequences $(-n), (-n^2)$ and (-2^n) diverge to $-\infty$.

Theorem 4:

If $(a_n) \to \infty$ and $a_n \neq 0$ for all $n \in$ N then $(1/a_n) \to 0$.

Proof:

Let $\varepsilon > 0$ be given. Since $(a_n) \to \infty$, there exists $m \in N$ such that $a_n > 1/\varepsilon$ for all $n \geq m$.

$$\therefore \frac{1}{a_n} < \varepsilon \text{ for all } n \geq m$$
$$\therefore \left|\frac{1}{a_n}\right| < \varepsilon \text{ for all } n \geq m$$
$$\therefore (1/a_n) \to 0$$

Note. The converse of the above theorem is not true. For example, consider the sequence (a_n) where $a_n = \frac{(-1)^n}{n}$. Clearly $(a_n) \to 0$.

Now $\left(\frac{1}{a_n}\right) = \left(\frac{n}{(-1)^n}\right) = -1, 2, -3, 4, \ldots..$ which neither converges nor diverges to ∞ or $-\infty$.

Thus if a sequence $(a_n) \to 0$, then the sequence $(1/a_n)$ need not converge or diverge.

Theorem 5:

If $(a_n) \to 0$ and $a_n > 0$ for, all $n \in$ N, then $(1/a_n) \to \infty$.

Proof:

Let $k > 0$ be any given real number. Since $(a_n) \to 0$ there exists $m \in N$ such that $|a_n| < 1/k$ for all $n \geq m$..

$$\therefore a_n < 1/k \text{ for all } n \geq m \text{ (since } a_n > 0)$$
$$\therefore 1/a_n > k \text{ for all } n \geq m.$$
$$\therefore (1/a_n) \to \infty.$$

Theorem 6:

Any sequence (a_n) diverging to ∞ is bounded below but not bounded above.

Proof:

Let $(a_n) \to \infty$. Then for any given real number $k > 0$ there etiof $m \in N$ such that $a_n > k$ for all $n \geq m$.

$\therefore$ k is not an upper bound of the sequence (a_n).

$\therefore$ (a_n) is not bounded above.

Now let $l = \min\{a_1, a_2, \ldots.., a_m, k\}$.

From (1) we see that $a_n \geq l$ for all n.

$\therefore$ (a_n) is bounded below.

Theorem 7:

Any sequence (a_n) diverging to $-\infty$ is bounded above but not below.

Proof is similar to that of Theorem 6.

Note:

1. The converse of the above theorem is not true. For example, 位 function, $f: \mathbf{N} \to \mathbf{R}$ defined by

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{2}n & \text{if } n \text{ is even} \end{cases}$$

determines the sequence 0,1,0,2,0,3, ... which is bounded below ad not bounded above. Also for any real number $k > 0$, we cannot find a natual number m such that $a_n > k$ for all $n \geq m$.

Hence this sequence does not diverge to ∞.

Similarly $f: \mathbf{N} \to \mathbf{R}$ given by $f(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{1}{2}n & \text{if } n \text{ is even} \end{cases}$ determines the sequence $0, -1, 0, -2, 0, \ldots$. which is bounded above asd not bounded below. However this sequence does not diverge to $-\infty$.

2. By theorem 2 any convergent sequence is bounded. Hence by theorem 6 we see that any convergent sequence cannot diverge to ∞. Similarly by theorem 7 it cannot diverge to $-\infty$. Also any sequence diverging to ∞ cannot converge or diverge to $-\infty$ and any sequence diverging to $-\infty$ cannot converge or diverge to ∞. Thus the three behaviours of a sequence namely convergence, divergence to ∞ and divergence to $-\infty$ are mutually exclusive.

However these three types of behaviour of sequences are not exhaustive since there exist sequences which neither converge nor diverge to ∞ nor diverge to $-\infty$.

Definition:

A sequence (a_n) which is neither convergent nor divergent to ∞ of $-\infty$ is said to be an oscillating sequence. An oscillating sequence which is bounded is said to be finitely oscillating. An oscillating sequence which is unbounded is said to be infinitely oscillating.

Examples.

1. Consider the sequence $((-1)^n)$. Since this sequence is bounded it cannot diverge to ∞ or $-\infty$ (by theorems 6 and 7). Also this sequence is not convergent (by example 5 of 1.5). Hence $((-1)^\kappa)$ is a finitely oscillating sequence.

2. The function $f: \mathbf{N} \rightarrow \mathbf{R}$ defined by

$$f(n) = \begin{cases} \frac{1}{2}n \text{ if } n \text{ is even} \\ \frac{1}{2}(1-n) \text{ if } n \text{ is odd} \end{cases}$$

determines the sequence $0,1,-1,2,-2,3,$ ..... . The range of this sequence is **Z**. Hence the sequence is neither bounded below nor bounded above. Hence it cannot converge or diverge to $\pm\infty$. This sequence is infinitely oscillating.

Exercises.

1. Discuss the behaviour of each of the following sequences.

   (a) $(n!)$

   (b) $1, \frac{1}{2}, 2, \frac{1}{3}, 3, \dots \dots, \frac{1}{n}, n, \dots \dots$

   (c) $((-1)^n 5)$

   (d) $((-1)^n + 5)$

   (e) $(-n^2)$

   (f) $(\sqrt{n})$

   (g) $(\cos n\pi)$

   (b) $(\sin n\pi/2)$.

2. Show that if (a_n) diverges to $-\infty$ and $a_n \neq 0$ for all n, then $(1/2$ converges to 0 .

3. If $(a_n) \to 0$ and $a_n < 0$ for all n prove that $(1/a_n) \to -\infty$.

1.7. The Algebra of Limits:

In this section we prove a few simple theorems for sequences vied are very useful in calculating limits of sequences.

Theorem 8:

If $(a_n) \to a$ and $(b_n) \to b$ then $(a_n + b_n) \to a + b$.

Proof:

Let $\varepsilon > 0$ be given.

Now $|a_n + b_n - a - b| = |a_n - a + b_n - b|$
$\leq |a_n - a| + |b_n - b|$ ………….(1)

Since $(a_n) \to a$, there exists a natural number n_1 such that

$|a_n - a| < \frac{1}{2}\varepsilon$ for all $n \geq n_1$ ………….. (2)

Since $(b_n) \to b$, there exists a natural number n_2 such that

$|b_n - b| < \frac{1}{2}\varepsilon$ for all $n \geq n_2$ ………….. (3)

Let $m = \max\{n_1, n_2\}$.

Then $|a_n + b_n - a - b| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$ for all $n \geq m$.

(by 1,2 and 3)

$\therefore (a_n + b_n) \to a + b$

Note:

Similarly we can prove that $(a_n - b_n) \to a - b$.

Theorem 9:

If $(a_n) \to a$ and $k \in \mathbf{R}$ then $(ka_*) \to ka$.

Proof:

If $k = 0, (ka_n)$ is the constant sequence 0,0,0, and hence the result is trivial.

Now, let $k = 0$.

Then $|ka_n - ka| = |k||a_n - a|$ ....(1)

Let $\varepsilon > 0$ be given.

Since $(a_n) \to a$, there exists $m \in$ N

such that $|a_n - a| < \frac{\varepsilon}{|k|}$ for all $n \geq m$. .... (2)

$\therefore |ka_n - ka| < \varepsilon$ for all $n \geq m$ by (1 and 2).

$\therefore (ka_n) \to ka$.

Theorem 10:

If $(a_n) \to a$ and $(b_n) \to b$ then $(a_n b_n) \to ab$.

Proof:

Let $\varepsilon > 0$ be given.

$$\begin{aligned} \text{Now, } |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &\leq |a_n b_n - a_n b| + |a_n b - ab| \\ &= |a_n||b_n - b| + |b||a_n - a| \quad \ldots\ldots\ldots (1) \end{aligned}$$

Also, since $(a_n) \to a, (a_n)$ is a bounded sequence. (by theorem 2)

$\therefore$ There exists a real number $k > 0$ such that $|a_n| \leq k$ for all n. ....(2)

Using (1) and (2) we get

$|a_n b_n - ab| \leq k|b_n - b| + |b||a_n - a|$ .... (3)

Now since $(a_n) \to a$ there exists a natural number n_1 such that

$|a_n - a| < \frac{\varepsilon}{2|b|}$ for all $n \geq n_1$ .... (4).

Since $(b_n) \to b$, there exists a natural number n_2 such that

$|b_n - b| < \frac{\varepsilon}{2k}$ for all $n \geq n_2$ .... (5)

Let $m = \max\{n_1, n_2\}$. Then

Let $m = \max\{n_1, n_2\}$

$|a_n b_n - ab| < k\left(\frac{\varepsilon}{2k}\right) + |b|\left(\frac{\varepsilon}{2|b|}\right) = \varepsilon$ for all $n \geq m$(by 3,4 and 5).

Hence $(a_n b_n) \to ab$.

Theorem 11:

If $(a_n) \to a$ and $a_n \neq 0$ for all n and $a = 0$, the $e_n \left(\frac{1}{a_n}\right) \to \frac{1}{a}$.

Proof:

Let $\varepsilon > 0$ be given.

We bave $\left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a_n - a}{a_n a}\right| = \frac{1}{|a_n||a|}|a_n - a|$ …………(1)

Now, $a = 0$. Hence $|a| > 0$.

Since $(a_n) \to a$ there exists $n_1 \in$ N such that $|a_n - a| < \frac{1}{2}|a|$ for all $n \geq n_1$.

Hence $|a_n| > \frac{1}{2}|a|$ for all $n \geq n_1$ ………….(2)

Using (1) and (2) we get

$\left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{2}{|a|^2}|a_n - a|$ for all $n \geq n_1$ ………… (3)

Now since $(a_n) \to a$ there exists $n_2 \in$ N such that

$|a_n - a| < \frac{1}{2}\varepsilon|a|^2$ for all $n \geq n_2$ ………….. (4)

Let $m = \max\{n_1, n_2\}$

$$\therefore \left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{2|a|^2\varepsilon}{|a|^2 2} = \varepsilon \text{ for all } n \geq m \quad (by\ 3\ and\ 4)$$

$$\therefore \left(\frac{1}{a_n}\right) \to \frac{1}{a}$$

Corollary:

Let $(a_n) \to a$ and $(b_*) \to b$ where $b_{\mathrm{n}} \neq 0$ for all n and $b = 0$.

Then $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$.

Proof:

$\left(\frac{1}{b_{\mathrm{n}}}\right) \to \frac{1}{b}$ (by theorem 11).

$\therefore \left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$ (by theorem 10).

Note:

Even if $\lim\limits_{x\to-} a_n$ and $\lim\limits_{n\to-} b_n$ do not exist, $\lim\limits_{x\to\infty}(a_n + b_n)$ and $\lim\limits_{n\to-} \frac{a_n}{b_n}$ may exist. For example let $a_n = ((-1)^n)$ and $b_n = ((-1)^{n+1})$. Clearly $\lim\limits_{n} a_n$, and $\lim\limits_{n\to-} b_n$ do not exist. Now $(a_n + b_n)$ is the constant sequence 0,0,0, ... Each of $(a_n b_n)$ and (a_n/b_n) is the constant sequence $-1, -1, \ldots\ldots$ Hence $(a_n + b_n) \to 0. (a_n, b_n) \to -1$ and $(a_n/b_n) \to -1$.

Theorem 12:

If $(a_n) \to a$ then $(|a_n|) \to |a|$.

Proof:

Let $\varepsilon > 0$ be given.

Now, $||a_n| - |a|| \leq |a_s - a|$ …………..(1)

Since $(a_n) \to a$, there exists $m \in$ N such that $|a_n - a| < \varepsilon$ for all $n \geq m$.

Hence from (1) we get $\big||a_n| - |a|\big| < \varepsilon$ for all $n \geq m$.

Hence $(|a_n|) \to |a|$.

Theorem 13:

If $(a_*) \to a$ and $a_n \geq 0$ for all n then $a \geq 0$.

Proof:

Suppose $a < 0$. Then $-a > 0$.

Cboose ε such that $0 < \varepsilon < -a$ so that $a + \varepsilon < 0$.

Now, since $(a_n) \to a$, there exists $m \in N$ such that $|a_0 - a| < \varepsilon$ forall $n \geq m$.

$\therefore a - \varepsilon < a_n < a + \varepsilon$ for all $n \geqslant m$

Now, since $a + \varepsilon < 0$, we have $a_\text{n} < 0$ for all $n \geq m$ which is a contadiction since $a_\text{n} \geq 0$.

Hence $a \geq 0$.

Note:

In the above theorem if $a_\text{n} > 0$ for all n, we cannot say that a > 0.

For example consider the sequence $\left(\frac{1}{n}\right)$. Here $\frac{1}{n} > 0$ for all n and $\left(\frac{1}{n}\right) \to 0$.

Theorem 14:

If $(a_n) \to a, (b_*) \to b$ and $a_n \leq b_n$ for all n, then $a \leq b$.

Proof:

Since $a_n \leq b_n$, we have $b_n - a_n \geq 0$ for all n.

Also $(b_* - a_*) \to b - a$ (by theorem 8)

$\therefore b - a \geq 0$(by theorem 13)

$\therefore a \leq b$.

Theorem 15:

If $(a_n) \to l, (b_n) \to l$ and $a_n \leq c_n \leq b_*$ for all n, $(c_\text{a}) \to l$.

Proof:

Let $\varepsilon > 0$ be given.

Since $(a_\text{N}) \to l$, there exists $n_1 \in$ N such that $l - \varepsilon < a_n < l + \varepsilon$, for all $n \geq n_1$.

Similarly, there exists $n_2 \in \mathbf{N}$ such that $l - \varepsilon < b_n < l + \varepsilon$ for all n_2

Let $m = \max\{n_1, n_2\}$.

$\therefore l - \varepsilon < a_n \leq c_n \leq b_n < l + \varepsilon$ forall $n \geq m$.
$\therefore l - \varepsilon < c_n < l + \varepsilon$ for all $n \geq m$.
$\therefore |c_n - l| < \varepsilon$ for all $n \geq m$.
$\therefore (c_n) \to l$.

Theorem 16:

If $(a_n) \to a$ and $a_n \geq 0$ for all n and $a \neq 0$, *then* $(\sqrt{a_n}) \to \sqrt{a}$.

Proof:

Since $a_n \geq 0$ for all $n, a \geq 0$. (by theorem 13)

Now, $|\sqrt{a_n} - \sqrt{a}| = \left|\frac{a_n - a}{\sqrt{a_n} + \sqrt{a}}\right|$

Since $(a_n) \to a = 0$, as in theorem 11 we obtain $a_* > \frac{1}{2}a$ for all $n \geq n_1$

$\therefore \sqrt{a_4} > \sqrt{\left(\frac{1}{2}a\right)}$ for all $n \geq n_1$.

$\therefore |\sqrt{a_n} - \sqrt{a}| < \frac{\sqrt{2}}{(\sqrt{2}+1)\sqrt{a}}|a_n - a|$ for all $n \geq n_1$ ……….. (1)

Now, let $\varepsilon > 0$ be given.

Since $(a_n) \to a$, there exists $n_2 \in$ N such that

$|a_n - a| < \varepsilon\sqrt{a}(\sqrt{2} + 1)/\sqrt{2}$ for all $n \geq n_2$ ………….(2)

Let $m = \max\{n_1, n_2\}$.

Then $|\sqrt{a_n} - \sqrt{a}| < \varepsilon$ for all $n \geq m$ (by 1 and 2).

$\therefore (\sqrt{a_n}) \to \sqrt{a}$

Theorem 17:

If $(a_n) \to \infty$ and $(b_n) \to \infty$ then $(a_n + b_n) \to \infty$.

Proof:

Let $k > 0$ be any given real number.

Since $(a_n) \to \infty$, there exists $n_1 \in$ N such that $a_n > \frac{1}{2}k$ for all $n \geq n_1$.

Similarly, there exists $n_2 \in$ N such that $b_n > \frac{1}{2}k$ for all $n \geq n_2$.

Let $m = \max\{n_1, n_2\}$.

Then $a_n + b_n > k$ for all $n \geq m$.

$\therefore (a_n + b_n) \to \infty$

Theorem 18:

If $(a_n) \to \infty$ and $(b_n) \to \infty$ then $(a_n b_n) \to \infty$.

Proof:

Let $k > 0$ be any given real number.

Since $(a_n) \to \infty$, there exists $n_1 \in$ N such that $a_n > \sqrt{k}$ for all $n \geq n_1$.

Similarly there exists $n_2 \in$ N such that $b_n > \sqrt{k}$ for all $n \geq n_2$..

Let $m = \max\{n_1, n_2\}$.

Then $a_n b_n > k$ for all $n \geq m$.

$\therefore (a_n b_n) \to \infty$

Theorem 19:

Let $(a_n) \to \infty$. Then

(i) if $c > 0, (ca_n) \to \infty$.

(ii) if $c < 0, (ca_n) \to -\infty$.

Proof:

(i) Let $c > 0$. Let $k > 0$ be any given real number.

Since $(a_n) \to \infty$, there exists $m \in N$ such that $a_n > k/c$ for all $n \geq m$

$\therefore ca_n > k$ for all $n \geq m$

$\therefore (ca_n) \to \infty$

(ii) Let $c < 0$. Let $k < 0$ be any given real number. Then $k/c > 0$,

$\therefore$ There exists $m \in$ N such that $a_n > k/c$ for all $n \geq m$.

$\therefore$ $ca_n < k$ for all $n \geq m$ (since $c < 0$).

$\therefore$ $(ca_a) \to -\infty$.

Theorem 20:

If $(a_n) \to \infty$ and (b_n) is bounded then $(a_n + b_n) \to \infty$.

Proof:

Since (b_s) is bounded, there exists a real number $m < 0$ such ithat

$b_n > m$ for all n. ………….(1)

Now, let $k > 0$ be any real number.

Since $m < 0, k - m > 0$.

Since $(a_n) \to \infty$, there exists $n_0 \in N$ such that

$a_n > k - m$ for all $n \geq n_0$ ……………..(2)

$\because$ $a_n + b_n > k - m + m = k$ for all $n \geq n_0$ (by 1 and 2).

$\therefore$ $(a_n + b_n) \to \infty$.

Problem 1.

Show that $\lim_{n \to -} \frac{3n^2+2n+5}{6n^2+4n+7} = \frac{1}{2}$.

Solution:

$$a_n = \frac{3n^2+2n+5}{6n^2+4n+7} = \frac{3+\frac{2}{n}+\frac{5}{n^2}}{6+\frac{4}{n}+\frac{7}{n^2}}.$$

Now, $\lim_{x \to -} \left(3 + \frac{2}{n} + \frac{5}{n^2}\right) = 3 + 2 \lim_{n \to -\infty} \frac{1}{n} + 5 \lim_{x \to -n^2} \frac{1}{2}$.

$$= 3 + 0 + 0 = 3$$

Similarly, $\lim_{n \to \infty} \left(6 + \frac{4}{n} + \frac{7}{n^2}\right) = 6$.

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\left(3+\frac{2}{n}+\frac{5}{n^2}\right)}{\left(6+\frac{4}{n}+\frac{7}{n^2}\right)}$$

$$= \frac{\lim_{n\to\infty}\left(3+\frac{2}{n}+\frac{5}{n^2}\right)}{\lim_{n\to\infty}\left(6+\frac{4}{n}+\frac{7}{n^2}\right)}$$

$$= \frac{3}{6} = \frac{1}{2}.$$

Problem 2:

Show that $\lim_{n\to\infty}\left(\frac{1^2+2^2+\cdots\ldots..+n^2}{n^3}\right) = \frac{1}{3}$.

Solution:

We know that $1^2+2^2+\cdots.+n^2 = \frac{n(n+1)(2n+1)}{6}$.

$$\therefore \lim_{n\to-\infty} \frac{1^2+2^2+\cdots.+n^2}{n^3} = \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}$$

$$= \lim_{n\to-\infty}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= \frac{1}{3}$$

Problem 3:

Show that $\lim_{n\to\infty}\frac{n}{\sqrt{(n^2+1)}} = 1$.

Solution:

$$\lim_{n\to--}\frac{n}{\sqrt{(n^2+1)}} = \lim_{n\to-\infty}\frac{1}{\sqrt{\left(1+\frac{1}{n^2}\right)}}$$

$$= \frac{1}{\lim_{n\to-\infty}\sqrt{\left(1+\frac{1}{n^2}\right)}} \text{(by theorem 11)}$$

$$= \frac{1}{\sqrt{\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)}} \text{(by theorem 16)}$$

$$= 1$$

Problem 4:

Show that if $(a_n) \to 0$ and (b_n) is bounded, then $(a_n b_n) \to 0$.

Solution:

Since (b_n) is bounded, there exists $k > 0$ such that $|b_n|_{k_k}$ all n.

$\therefore |a_n b_n| \le k|a_n|$.

Now, let $\varepsilon > 0$ be given.

Since $(a_n) \to 0$, there exists $m \in$ N such that $|a_n| < \varepsilon/k$ for all n_{γ_1}

$\therefore |a_n b_n| < \varepsilon$ for all $n \ge m$.

$\therefore (a_n b_n) \to 0$.

Problem 5:

Show that $\lim_{n\to-\infty} \frac{\sin n}{n} = 0$.

Solution:

$|\sin n| \le 1$ for all n.

$\therefore (\sin n)$ is a bounded sequence.

Also, $\left(\frac{1}{n}\right) \to 0$

$\therefore \left(\frac{\sin n}{n}\right) \to 0$ (by problem 4$\Big)$

Problem 6:

Show that $\lim_{n\to-\infty} \left(a^{1/n}\right) = 1$ where $a > 0$ is any real muntrat.

Solution:

Case (i)

Let $a = 1$. Then $a^{1/n} = 1$ for each n.

Hence $\left(a^{1/n}\right) \to 1$

Case (ii)

Let $a > 1$. Then $a^{1/n} > 1$.

Let $a^{1/n} = 1 + h_n$ where $h_n > 0$.

$$\therefore a = (1 + h_n)^n$$
$$= 1 + nh_n + \cdots \ldots + h_n^n$$
$$> 1 + nh_n.$$
$$\therefore h_n < \frac{a-1}{n}.$$
$$\therefore 0 < h_n < \frac{a-1}{n}.$$

Hence $\lim_{n \to -\infty} h_n = 0$.

$\therefore \left(a^{1/n}\right) = (1 + h_n) \to 1$

Case (iii)

Let $0 < a < 1$. Then $1/a > 1$.

$$\therefore (1/a)^{\frac{1}{n}} \to 1 \text{ (by case (ii))}.$$
$$\therefore \left(\frac{1}{a^{\frac{1}{n}}}\right) \to 1.$$

$\therefore \left(a^{1/n}\right) \to 1$ (by theorem 11)

Problem 7:

Show that $\lim\left(n^{1/n}\right) = 1$.

Solution:

Clearly $n^{1/n} \geq 1$ for all n.

Let $n^{1/n} \geq 1 + h_n$ where $h_n \geq 0$.

Then $n = (1 + h_n)^n$

$= 1 + nh_n + nc_2h_n^2 + \cdots \ldots + h_n^n$

$> \frac{1}{2}n(n-1)h_n^2$

$\therefore h_n^2 < \frac{2}{n-1}$.

$\therefore h_n < \sqrt{\frac{2}{n-1}}$.

Since $\sqrt{\frac{2}{n-1}} \to 0$ and $h_n \geq 0, (h_n) \to 0$.

$\therefore (n^{1/n}) = (1 + h_n) \to 1$

Problem 8.

Show that $\lim_{n\to-\infty}\left(\frac{1}{\sqrt{(2n^2+1)}} + \frac{1}{\sqrt{(2n^2+2)}} + \cdots \ldots + \frac{1}{\sqrt{(2n^2+n)}}\right) = \frac{1}{\sqrt{2}}$

Solution:

Let $a_n = \frac{1}{\sqrt{(2n^2+1)}} + \frac{1}{\sqrt{(2n^2+2)}} + \cdots + \frac{1}{\sqrt{(2n^2+n)}}$

Then we have the inequality $\frac{n}{\sqrt{(2n^2+n)}} \leq a_n \leq \frac{n}{\sqrt{(2n^2+1)}}$.

$$\therefore \frac{1}{\sqrt{\left(2+\frac{1}{n}\right)}} \leq a_n \leq \frac{1}{\sqrt{\left(2+\frac{1}{n^2}\right)}}$$

Now, $\lim_{n\to-\infty} \frac{1}{\sqrt{\left(2+\frac{1}{n}\right)}} = \lim_{n\to-\infty} \frac{1}{\sqrt{\left(2+\frac{1}{n^2}\right)}} = \frac{1}{\sqrt{2}}$.

$\therefore \lim_{n\to\infty} a_n = \frac{1}{\sqrt{2}}$ (by theorem 15).

Problem 9:

Give an example to show that if (a_n) is a sequence diverging ∞ and (b_n) is a sequence diverging to $-\infty$ then $(a_n + b_n)$ need not be: divergent sequence.

Solution:

Let $(a_n) = (n)$ and $(b_n) = (-n)$.

Clearly $(a_n) \to \infty$ and $(b_n) \to -\infty$.

However $(a_n + b_n)$ is the constant sequence $0,0,0,\ldots\ldots$ which converges to 0 .

Exercises.

1.Evaluate the limits of the following sequences as $n \to \infty$.

(a) $\left(\frac{3n-4}{2n+7}\right)$ (b) $\left(\frac{4-2n+6n^2}{7-6n+9n^2}\right)$ (c) $\left(\frac{(n^2+3)(n^3+9)}{(n+1)(n^4+6)}\right)$

(d) $\left(\sqrt{(n^2+n)} - n\right)$ (e) $\frac{\sqrt{(3n^2-5n+4)}}{2n-7}$ (f) $\left(\frac{n^2+n+1}{n^3+2}\right)$

(g) $\left(\frac{1+2+3+\cdots\ldots\ldots+n}{n^2}\right)$ (h) $((-1)^n/n)$ (h) $\frac{n^2}{\sqrt{(n^4+3n^2+1)}}$

2. A sequence (a_n) is called a null sequence if $(a_n) \to 0$. Show that if (a_n) and (b_n) are null sequences then $(a_n + b_n), (a_n b_n), (ka_n)$ and $(\,|a_n|)$ are also null sequences.

3. If $(a_n) \to -\infty$ and $(b_n) \to -\infty$, then show that $(a_n + b_n) \to -\infty$ and $(a_n b_n) \to \infty$.

4. If $(a_n) \to -\infty$, then show that $(ka_n) \to -\infty$ if $k > 0$ and $(ka_n) \to \infty$ if $k < 0$.

5. If $(a_n) \to -\infty$ and (b_n) is a bounded sequence then show that $(a_i + b_n) \to -\infty$.

6. Show that following sequences diverge to ∞.

(a) $(n^3 + n^2 + n + 1)$

(b) $(n + (-1)^n/n^2)$

(c) (n^n)

(d) $\left(\frac{n^2+3n+1}{n+1}\right)$ (Hint : $\frac{n^2+3n+1}{n+1} = n + 2 - \frac{1}{n+1}$).

7. Prove the following.

(a) $\lim_{n\to-}\left(\frac{1}{\sqrt{(n^2+1)}} + \frac{1}{\sqrt{n^2+2}} + \cdots\ldots + \frac{1}{\sqrt{(n^2+n)}}\right) = 1.$

(b) $\lim_{n\to\infty}\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots . + \frac{1}{(2n)^2}\right) = 0.$

(c) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{(n+1)}} + \cdots . + \frac{1}{\sqrt{(2n)}}\right) = \infty.$

8. Give examples of sequences (a_n) and (b_n) such that

(a) $(a_n) \to \infty, (b_n) \to \infty$ and $(a_n - b_n)$ converges.

(b) $(a_n) \to \infty, (b_n) \to \infty$ and $(a_n - b_n)$ converges to 5 .

(c) $(a_n) \to \infty, (b_n) \to \infty$ and $(a_n - b_n) \to \infty$.

Unit II

Behaviour of Monotonic Sequences – Some theorem on limits – Sub sequences – Limit points – Cauchy sequences.

Chapter 2: Sections 2.1 – 2.5.

2.1. Behaviour of Monotonic Sequences:

The following theorem gives the complete behaviour of monotonic sequences.

Theorem 1:

(i) A monotonic increasing sequence which is bounded above converges to its. 1.u.b.

(ii) A monotonic increasing sequence which is not bounded above diverges to ∞

(iii) A monotonic decreasing sequence which is bounded below converges to its g.l.b.

(iv) A monotonic decreasing sequence which is not bounded below diverges to $-\infty$.

Proof:

(i)Let (a_n) be a monotonic increasing sequence which is bounded above.

Let k be the l, u, b of the sequence.

Then $a_n \leq k$ for all n. …………..(1)

Now, let $\varepsilon > 0$ be given.

$\therefore$ $k - \varepsilon < k$ and hence $k - \varepsilon$ is not an upper bound of (a_n).

Hence, there exists a_m such that $a_m > k - \varepsilon$.

Now, since (a_n) is monotonic increasing, $a_n \geq a_m$ for all $n \geq m$.

Hence $a_n > k - \varepsilon$ for all $n \geq m$ …………(2)

$\therefore$ $k - \varepsilon < a_n \leq k$ for all $n \geq m$ (by 1 and 2)

$\therefore$ $|a_n - k| < \varepsilon$, for all $n \geq m$.

$\therefore$ $(a_n) \to k$.

(ii) Let (a_n) be a monotonic increasing sequence which is not bounded above.

Let $k > 0$ be any real number.

Since (a_n) is not bounded, there exists $m \in$ N such that $a_m > k$.

Also $a_n \geq a_m$ for all $n \geq m$.

$\therefore a_n > k$ for all $n \geq m$.
$\therefore (a_n) \to \infty$.

Proof of (iii) is similar to that of (i).

Proof of (iv) is similar to that of (ii).

Note:

The above theorem shows that a monotonic sequence cither converges or diverges. Thus a monotonic sequence cannot be an oscillating sequence.

Problem 1:

Let $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots . + \frac{1}{n!}$ Show that $\lim_{n\to\infty} a_n$ exists and lies between 2 and 3 .

Solution:

Clearly (a_n) is a monotonic increasing sequence.

$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

$$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \ldots + \frac{1}{2^{n-1}}$$

Also,

$$= 1 + \left(\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \right)$$

$$= 1 + 2\left(1 - \frac{1}{2^n}\right)$$

$$= 3 - \frac{1}{2^{n-1}} < 3$$

$\therefore a_n < 3$.

$\therefore (a_n)$ is bounded above.

$\therefore \lim a_n$ exists.

Also $2 < a_n < 3$ for all n.

$\therefore 2 \leq \lim a_n \leq 3.$

Hence the result.

Note:

The limit of the above sequence is denoted by e.

Problem 2:

Show that the sequence $\left(1+\frac{1}{n}\right)^n$ converges.

Solution:

Let $a_n = \left(1+\frac{1}{n}\right)^n$

By binomial theorem,

$$a_n = 1 + 1 + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots .. + \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) +$$

$$\dots\dots\dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{n-1}{n}\right)$$

$$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots .. + \frac{1}{n!}$$

< 3 (refer problem 1).

$\therefore (a_n)$ is bounded above.

Also,

$$a_{n+1} = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots \dots ..$$

$$\dots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\dots\dots\left(1-\frac{n}{n+1}\right)$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots$$

$$\dots\dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{n-1}{n}\right).$$

$\therefore a_{n+1} > a_n$

$\therefore (a_n)$ is monotonic increasing,

$\therefore (a_n)$ is a convergent sequence.

Problem 3:

Show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{1!}+\cdots\ldots+\frac{1}{n!}\right) = e$

Solution:

Let $a_n = \left(1+\frac{1}{n}\right)^n$ and $b_n = 1+\frac{1}{1!}+\cdots.+\frac{1}{n!}$.

Then $a_n < b_n$ for all n (refer problem 2 above).

$\therefore \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$ …………..(1)

Now, let $m > n$.

$$\begin{aligned} a_m &= \left(1+\frac{1}{m}\right)^m \\ &= 1+1+\frac{1}{2!}\left(1-\frac{1}{m}\right)+\frac{1}{3!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)+\cdots. \\ &+\frac{1}{n!}\left(1-\frac{1}{m}\right)\cdots\left(1-\frac{n-1}{m}\right)+\cdots.+\frac{1}{m!}\left(1-\frac{1}{m}\right)\cdots..\left(1-\frac{m-1}{m}\right) \\ &> 1+1+\frac{1}{2!}\left(1-\frac{1}{m}\right)+\cdots\ldots+\frac{1}{n!}\left(1-\frac{1}{m}\right)\cdots\ldots.\left(1-\frac{n-1}{m}\right) \end{aligned}$$

Fixing n and taking limit as $m\to\infty$ we get

$$\lim_{m\to\infty} a_m \ge 1+1+\frac{1}{2!}+\cdots\ldots+\frac{1}{n!} = b_n$$

Now taking limit as $n\to\infty$ we get

$$\lim_{m\to\infty} a_m \ge \lim_{m\to\infty} b_n \quad \ldots\ldots\ldots\ldots.. (2)$$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = e(\text{ by (1) and (2)})$$

Problem 4:

Let $a_n = \frac{1}{n+1}+\frac{1}{n+2}+\cdots.+\frac{1}{n+n}$. Show that (a_n) converges.

Solution:

$a_{n+1} - a_n$

$$= \left(\frac{1}{n+2} + \cdots \ldots + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \cdots \ldots + \frac{1}{n+n}\right)$$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}.$$

$$= \frac{1}{2n+1} - \frac{1}{2n+2} > 0 \text{ for all } n$$

$\therefore a_{n+1} > a_n$ for all n.

$\therefore (a_n)$ is a monotonic increasing sequence.

Also $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots \ldots + \frac{1}{n+n}$.

$< \frac{1}{n} + \frac{1}{n} + \cdots \ldots + \frac{1}{n} = 1$ for all n.

$\therefore (a_n)$ is bounded above.

$\therefore (a_n)$ converges.

Problem 5:

Let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \quad + \frac{1}{n}$. Show that (a_n) diverges to ∞.

Solution:

Clearly (a_n) is a monotonic increasing sequence:

Now, let $m = 2^n - 1$

$$a_m = 1 + \frac{1}{2} + \cdots \ldots + \frac{1}{2^n - 1}$$

$$= 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \cdots . + \left(\frac{1}{2^{n-1}} + \cdots . . + \frac{1}{2^n - 1}\right)$$

$$> 1 + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \ldots \ldots \left(\frac{1}{2^n} + \cdots \ldots \ldots . . + \frac{1}{2^n}\right)$$

$$= 1 + (n-1)\frac{1}{2} = \frac{1}{2}(n+1)$$

$$\therefore a_m > \frac{1}{2}(n+1)$$

$\therefore (a_n)$ is not bounded above. Hence $(a_n) \to \infty$.

Problem 6:

Prove that $\left(\frac{n!}{n^n}\right)$ converges.

Solution:

Let $a_n = \frac{n!}{n^n}$.

Then $\frac{a_n}{a_{n+1}} = \frac{n!}{n^n}\frac{(n+1)^{n+1}}{(n+1)!} = \left(\frac{n+1}{n}\right)^n > 1$.

$\therefore a_n > a_{n+1}$ forall $n \in$ N.

$\therefore (a_n)$ is a monotonic decreasing sequence.

Also $a_n > 0$ for all $n \in N$.

$\therefore (a_n)$ is bounded below.

$\therefore (a_n)$ converges.

Problem 7:

Discuss the behaviour of the geometric sequence (r^n) .

Solution:

Case (i) Let $r = 0$.

Then (r^n) reduces to the constant sequence $0,0, \dots \dots \dots$ and hence converges to 0.

In this case (r^n) reduces to the constant sequence $1,1,1, \dots \dots$ and hence converges to 1.

In this case, (r^n) is a monotonic decreasing sequence and $(r^n) > 0$

$\therefore (r^n)$ is monotonic decreasing and bounded below and hence (r^n) converges.

Let $(r^n) \to l$

Since $r^n > 0$ for all $n, l > 0$. ……………(1)

We claim that $l = 0$.

Let $\varepsilon > 0$ be given. Since $(r^n) \to l$, there exists $m \in N$ such that

$1 < r^n < l + \varepsilon$ for all $n \geq m$.

Fix $n > m$. Then $l < r^{n+1}$ …………(2)

Also $r^{n+1} = r.r^n < r(l+\varepsilon)$. ....(3)

$\therefore l < r(l+\varepsilon)$ (by 2 and 3).

$\therefore 1 < \left(\frac{r}{1-r}\right)\varepsilon$.

Since this is true for every $\varepsilon > 0$, we get $l \leq 0$. ....(4)

$\therefore l = 0$ (by 1 and 4).

Case(iv) Let $-1 < r < 0$.

Then $r^n = (-1)^n |r|^n$ where $0 < |r| < 1$. =

By cast (iii) $(|r|^n) \to 0$.

Also $((-1)^n)$ is a bounded sequencé.

$\therefore ((-1)^n |r|^n)$ converges to 0 (by problem 4 of 3.6)

$\therefore (r^n) \to 0$.

Case (v) Let $r = -1$.

In this case (r^n) reduces to $-1, 1, -1$, which oscillates finitely.

Case (vi) Let $r > 1$.

Then $0 < \frac{1}{r} < 1$ and hence $\left(\frac{1}{r^n}\right) \to 0$ (by case (iii))

$\therefore (r^n) \to \infty$: (by theorem 5 of 1.5)

Case (vii) Let $r < -1$.

Then the terms of the sequence (r^n) are alternatively positive and negative. Also $|r| > 1$ and hence by case (vi) $(|r|^\mu)$ is unbounded.

$\therefore (r^n)$ oscillates infinitely.

Thus (i) (r^*) converges if $-1 < r \leq 1$.

(ii) (r^n) diverges if $r > 1$.

(iii) (r'') oscillates if $r \leq -1$.

Problem 8:

Show that if $|r| < 1$ then $(nr^n) \to 0$.

Solution. The result is trivial if $r = 0$.

Let $0 < |r| < 1$. Then $|r| = \dfrac{1}{1+p}, p > 0$.

$$\therefore |r|^n = \frac{1}{(1+p)^n}$$

$$= \frac{1}{1 + np + \frac{n(n-1)}{1.2}p^2 + \cdots \ldots..}$$

$$< \frac{2}{n(n-1)p^2}$$

$$\therefore |nr^n| < \frac{2}{(n-1)p^2}$$

Now, let $\varepsilon > 0$ be given.

Then $\dfrac{2}{(n-1)p^2} < \varepsilon$ provided $n > 1 + \dfrac{2}{p^2\varepsilon}$

$\therefore |nr^n| < \varepsilon$ if $n > 1 + \dfrac{2}{p^2\varepsilon}$.

$\therefore \lim\limits_{n\to\infty} nr^n = 0$

Problem 9:

Show that $\lim\limits_{n\to\infty} \frac{\log n}{n^p} = 0$ if $p > 0$.

Solution:

We have $e^p > 1$ (since $e > 1$)

$$\therefore \frac{1}{e^p} < 1$$

$\therefore \left(\frac{n}{(e^p)^n}\right) \to 0$ (by problem 8).

$\therefore$ Given $\varepsilon > 0$, there exists a natural number m such that

$\frac{n}{e^n} < \frac{\varepsilon}{e^p}$ for all $n \geq m$.

Now, let g be the positive integer such that $g \leq \log n < (g+1)$.

$$\therefore \frac{\log n}{n^p} < \frac{g+1}{n^p}$$

$$\leq \frac{g+1}{(e^g)^p} \text{ (since } e^g \leq n \text{ by (2))}$$

$$= \frac{e^p(g+1)}{e^{p(g+1)}}$$

$$< e^p \left(\frac{\varepsilon}{e^p}\right) \text{ provided } g+1 \geq m \text{ (using 1)}$$

$$\therefore \frac{\log n}{n^p} < \varepsilon \text{ provided } g+1 \geq m.$$

Now, if $n \geq e^m$, then $\log n \geq m$.

But $g + 1 > \log n$ (by (2))

$$\therefore n \geq e^m \Rightarrow g+1 \geq m.$$

$$\therefore \frac{\log n}{n^p} < \varepsilon \text{ provided } n \geq e^m.$$

$$\therefore \lim_{n\to\infty} \frac{\log n}{n^p} = 0.$$

Problem 10:

Let (a_n) and (b_n) be two sequences of positive terms such that $a_{n+1} = \frac{1}{2}(a_n + b_n)$ and $b_{n+1} = \sqrt{(a_n b_n)}$. Prove that (a_n) and (b_n) converge to the same limit.

Solution:

By hypothesis, a_{n+1} and b_{n+1} are respectively the A.M. and $C_{.M}$ between a_n and b_n.

Also we know that A.M. ≥ G.M.

Hence $a_{n+1} \geq b_{n+1}$ ....(1)

Moreover the A.M. and G.M. of two numbers lie between the w_0 numbers.

$\therefore a_n \geq a_{n+1} \geq b_n$ for all $n \in$ N. ....(2)

and $a_n \geq b_{n+1} \geq b_n$ for all $n \in$ N. ....(3)

$\therefore a_n \geq a_{n+1} \geq b_{n+1} \geq b_n$ for all $n \in$ N. (by 2 and 3)

$\therefore (a_n)$ is a monotonic decreasing sequence and (b_n) is a monotonic increasing sequence.

Further, $a_n \geq b_n \geq b_1$ for all $n \in$ N.

and $b_n \leq a_n \leq a_1$ for all $n \in$ N.

$\therefore$ (a_n) is a monotonic decreasing sequence bounded below by b_1 and (b_n) is a monotonic increasing sequence bounded above by a_1.

$\therefore (a_n) \to l($ say $)$ and $(b_n) \to m$ (say)

Now, $a_{n+1} = \frac{1}{2}(a_n + b_n)$.

Taking limit as $n \to \infty$, we get $l = \frac{1}{2}(l + m)$.

$\therefore l = m$.

Problem 11:

Let (a_n) be a sequence of positive terms such that $a_1 < a_2$ and $a_{n+2} = \frac{1}{2}(a_{n+1} + a_n)$. Then show that (a_{2n-1}) is a monotonic increasing sequence and (a_{2n}) is a decreasing sequence and both converge to limit.

Solution:

We have $a_{n+2} = \frac{1}{2}(a_{n+1} + a_n)$ and $a_1 < a_2$ …………..(1)

$\therefore a_3 = \frac{1}{2}(a_2 + a_1)$ and $a_1 < a_2$

$\therefore a_1 < a_3 < a_2$ … … … . . (2)

Also $a_4 = \frac{1}{2}(a_1 + a_2)$ and $a_1 < a_2$(by 1 and 2).

$\therefore a_3 < a_4 < a_2$ … … … … . . (3)

$\therefore a_1 < a_3 < a_4 < a_2$ (by 2 and 3)

Proceeding as above, we get $a_1 < a_3 < a_5 < a_6 < a_4 < a_2$ and so on.

$\therefore$ (d_{2n}) is a monotonic decreasing sequence bounded below by a_1 and (a_{2n-1}) is a monotonic increasing sequence bounded above by a_{s-} .

$\therefore (a_{2\alpha}) \to 1($ say $)$ and $(a_{2n-1}) \to m$ (say).

Now, $a_{2n+2} = \frac{1}{2}(a_{2n+1} + a_{2n})$ (by 1)

Taking limit as $n \to \infty$, we get $l = \frac{1}{2}(m + l)$.

$\therefore l = m.$

Now, let $\varepsilon > 0$ be given. Since $(a_{2\alpha}) \to l$, there exists $n \in$ N such that $|a_{2n} - l| < \varepsilon$ for all $n \geq n_1$.

Similarly there exists $n_2 \in$ N such that $|a_{2n-1} - l| < \varepsilon \cdot$ for all $n \geq n_2$. Let $m = \max\{n_1, n_2\}$

Then $|a_n - l| < \varepsilon$ for all $n \geq m$.

$\therefore (a_n) \to l$.

Now, $a_{n+2} = \frac{1}{2}(a_{n+1} + a_n)$

$a_{n+1} = \frac{1}{2}(a_n + a_{n-1})$.

……………

……………....

$a_4 = \frac{1}{2}(a_3 + a_2)$.

$a_3 = \frac{1}{2}(a_2 + a_1)$.

Adding, we get $a_{n+2} = \frac{1}{2}(a_1 + 2a_2)$.

Taking limit as $n \to \infty$, we get

$$l + \frac{1}{2}l = \frac{1}{2}(a_1 + 2a_2)$$

$l = \frac{1}{3}(a_1 + 2a_2)$.

Exercises:

1. Let (a_0) be a sequence of positive terms such that $a_1 < a_2$ and $a_{n+2} = \sqrt{(a_{n+1} a_n)}$. Then show that (a_{2+-}) is a monotonic increasing scquence and (a_2) is a monotonic decreasing sequence and both converge to the common limit $(a_1 a_2^2)^{1/3}$. Hence deduce that (a_2) converges to the same limit.

2. Let (a_N) be a sequence of positive terms such that $a_1 < a_2$ and $a_{n+2} = \frac{2a_{*-1}a_*}{a_{n+1}+a_*}$. Then show that (a_{2+-1}) is a monotonic increasing sequence and a_2, is a monotonic decreasing sequence and both converge to the common limit $\frac{a_1a_2}{3(2a_1+a_2)}$. Hence deduce that (a_n) converges to the same limit.

3. Verify whether the following sequences are monotonic and discuss tbeir behaviour.

   (i) $\left(\frac{2n-7}{3n+2}\right)$

   (ii) $\left(-\frac{1}{2n+1}\right)$

   (iii) $(\sqrt{(n+1)} - \sqrt{n})$

   (iv) $a_1 = 1$ and $a_{*+1} = \sqrt{(2+a_v)}$

4. Prove that $\left(\frac{an+d}{bn+c}\right)$ is a monotonic increasing or decreasing or a constant sequence according as $bd < ac, bd > ac, bd = ac$.

5. Show that the sequence whose n^{th} term is $\frac{x^n+n}{x^{n-1}+2n}$ converges to $\frac{1}{2}$ if $|x| \leq 1$ and converges to x if $|x| > 1$.

6. Show the sequence (a_2) given by $a_1 = \sqrt{2}$ and $a_{*,1} = \sqrt{(2a_2)}$ for all $n \geq 1$ converges to 2 .

2.2. Some Theorems on Limits:

Theorem 1: (Cauchy's first limit theorem)

If $(a_n) \to l$ then $\left(\frac{a_1 + a_2 + \cdots .. + a_n}{n}\right) \to l.$

Proof:

Case (i) Let $l = 0$.

Let$b_n = \frac{a_1 + a_2 + \cdots ... + a_n}{n}$

Let $\varepsilon > 0$ be given. Since, $(a_n) \to 0$ there exists $m \in N$

such that $|a_n| < \frac{1}{2}\varepsilon$ for all $n \geq m$. ....(1)

Now, let $n \geq m$.

$$\text{Then } |b_n| = \left|\frac{a_1 + a_2 + \cdots \ldots + a_n + a_{m+1} + \cdots \ldots + a_n}{n}\right|$$
$$\leq \frac{|a_1| + |a_2| + \cdots \ldots + |a_m|}{n} + \frac{|a_{m+1}| + \cdots \ldots + |a_n|}{n}$$
$$= \frac{k}{n} + \frac{|a_{m+1}| + \cdots \ldots + |a_n|}{n} \text{ where } k = |a_1| + \cdots \ldots + |a_m|$$
$$< \frac{k}{n} + \left(\frac{n-m}{n}\right)\frac{\varepsilon}{2} \text{ (by 1)}$$
$$< \frac{k}{n} + \frac{\varepsilon}{2}\left(\text{ since } \frac{n-m}{n} < 1\right) \ldots \ldots \ldots (2)$$

Now, since $\left(\frac{k}{n}\right) \to 0$, there exists $n_0 \in$ N such that

$\frac{k}{n} < \frac{1}{2}\varepsilon$ for all $n \geq n_0$ .... (3)

Let $n_1 = \max\{m, n_0\}$.

Then $|b_n| < \varepsilon$ for all $n \geq n_1$ (using 2 and 3).

$\therefore (b_n) \to 0$

Case (ii) Let $l \neq 0$.

Since $(a_n) \to l, (a_n - l) \to 0$.

$$\therefore \left(\frac{(a_1 - l) + (a_2 - l) + \cdots + (a_n - l)}{n}\right) \to 0 \text{ (by case i)}$$
$$\therefore \left(\frac{a_1 + a_2 \ldots \ldots + a_n - nl}{n}\right) \to 0$$
$$\therefore \left(\frac{a_1 + a_2 \ldots \ldots + a_n}{n} - l\right) \to 0.$$

$$\therefore \left(\frac{a_1 + a_2 \ldots \ldots + a_n}{n}\right) \to l$$

Note: The converse of the above theorem is not true. For example, coset. the sequence $(a_n) = ((-1)^n)$.

Then $b_n = \dfrac{a_1 + a_2 + \cdots . + a_n}{n} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{1}{n} & \text{if } n \text{ is odd} \end{cases}$.

Clearly $(b_n) \to 0$ and (a_n) is not convergent.

Theorem 2: (Cesaro's theorem)

If $(a_n) \to a$ and $(b_n) \to b$ then $\left(\frac{a_1b_n + a_2b_{n-1} + \cdots .. + a_nb_1}{n}\right) \to ab.$

Proof:

Let $c_n = \frac{a_1b_n + \cdots .... + a_nb_1}{n}.$

Now, put $a_n = a + r_n$ so that $(r_n) \to 0.$

Then $c_n = \frac{(a+r_1)b_n + \cdots ... + (a+r_n)b_1}{n}.$

$$= \frac{a(b_1 + \cdots ... + b_n)}{n} + \frac{r_1b_n + \cdots ... + r_nb_1}{n}$$

Now, by Cauchy's first limit theorem,

$$\left(\frac{b_1 + b_2 + \cdots ... + b_n}{n}\right) \to b.$$

$$\therefore \left(\frac{a(b_1 + b_2 + \cdots ... + b_n)}{n}\right) \to ab.$$

Hence it is enough if we prove that $\left(\frac{r_1b_n + \cdots ... + r_nb_1}{n}\right) \to 0.$

Now, since $(b_n) \to b, (b_n)$ is a bounded sequence. (by theorem 2 of 1.2)

$\therefore$ There exists a real number $k > 0$ such that $|b_n| \le k$ for all n.

$$\therefore \left|\frac{r_1b_n + \cdots ... + r_nb_1}{n}\right| \le k \left|\frac{r_1 + \cdots ... + r_n}{n}\right|$$

Since $(r_n) \to 0, \left(\frac{r_1 + \cdots ... + r_n}{n}\right) \to 0$ (by theorem 1)

$$\left(\frac{r_1b_n + \cdots .. + r_nb_1}{n}\right) \to 0$$

Hence the theorem.

Theorem 3: (Cauchy's second limit theorem)

Let (a_n) be a sequence of positive terms. Then $\lim\limits_{n\to\infty} a_n^{1/n} = \lim\limits_{a\to\infty} \frac{a_{n+1}}{a_a}$ provided the limit on the right hand side exists, whether finite or infinite.

Proof:

Case (i) $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, finite.

Let $t > 0$ be any given real number.

Then there exists $m \in$ N such that

$l - \frac{1}{2}\varepsilon < \frac{a_{n+1}}{a_n} < l + \frac{1}{2}\varepsilon$ for all $n \geq m$

Now choose n nIm.

Then$l - \frac{1}{2}\varepsilon < \frac{a_{m+1}}{a_m} < l + \frac{1}{2}\varepsilon$

$$l - \frac{1}{2}\varepsilon < \frac{a_{m+2}}{a_{m+1}} < l + \frac{1}{2}\varepsilon$$

....

……………………...

$$l - \frac{1}{2}\varepsilon < \frac{a_n}{a_{n-1}} < l + \frac{1}{2}\varepsilon$$

Multiplying these inequalities, we obtain

$$\left(l - \frac{1}{2}\varepsilon\right)^{n-m} < \frac{a_n}{a_n} < \left(l + \frac{1}{2}\varepsilon\right)^{n-m}$$

$$\therefore a_m \frac{\left(l - \frac{1}{2}\varepsilon\right)^n}{\left(l - \frac{1}{2}\varepsilon\right)^m} < a_n < a_m \frac{\left(l + \frac{1}{2}\varepsilon\right)^n}{\left(l + \frac{1}{2}\varepsilon\right)^m}$$

$\therefore k_1 \left(l - \frac{1}{2}\varepsilon\right)^n < a_n < k_2 \left(l + \frac{1}{2}\varepsilon\right)^n$ where k_1, k_2 are some consume

$$\therefore k_1^{1/n}\left(l - \frac{1}{2}\varepsilon\right) < a_n^{1/n} < k_2^{1/n}\left(l + \frac{1}{2}\varepsilon\right) \ldots\ldots\ldots\ldots(1)$$

Now, $\left(k_1^{1/n}\left(l - \frac{1}{2}\varepsilon\right)\right) \to l - \frac{1}{2}\varepsilon$ (since $\left(k_1^{1/n}\right) \to l.$).

(by solved problem 6 of 1.6)

$\therefore$ There exists $n_1 \in$ N such that

$$\left(l-\frac{1}{2}\varepsilon\right)-\frac{1}{2}\varepsilon < k_1^{1/n}\cdot\left(l-\frac{1}{2}\varepsilon\right) < \left(l-\frac{1}{2}\varepsilon\right)+\frac{1}{2}\varepsilon \text{ for all } n \geq n_1 \text{ ....(2)}$$

Similarly, there exists $n_2 \in$ N such that

$$\left(l+\frac{1}{2}\varepsilon\right)-\frac{1}{2}\varepsilon < k_2^{1/n}\left(l+\frac{1}{2}\varepsilon\right) < \left(l+\frac{1}{2}\varepsilon\right)+\frac{1}{2}\varepsilon \text{ for all } n \geq n_2 \text{........(3)}$$

Let $n_0 = \max\{m, n_1, n_2\}$.

Then $l-\varepsilon < k_1^{1/n}\left(l-\frac{1}{2}\varepsilon\right) < a_n^{1/n} < k_2^{1/n}\left(l+\frac{1}{2}\varepsilon\right) < l+\varepsilon$

for all $n \geq n_0$ (by 1,2 and 3)

$\therefore l-\varepsilon < a_n^{1/n} < l+\varepsilon$ for all $n \geq n_0$. Hence $\left(a_n^{1/n}\right) \to l$.

Case (ii) $\lim\limits_{n\to-\infty} \frac{a_{n+1}}{a_n} = \infty$.

Then $\lim\limits_{n\to-\infty} \frac{\left(\frac{1}{a_{n+1}}\right)}{\left(\frac{1}{a_n}\right)} = 0,$ (by theorem 3.4)

$\therefore$ By case (i), $\left(\frac{1}{a_n}\right)^{\frac{1}{n}} \to 0.$

$\therefore \left(a_n^{\frac{1}{n}}\right) \to \infty$ (by theorem 5 of 1.5).

Theorem 4:

Let (a_n) be any sequence and $\lim_{*\to} \left|\frac{a_*}{a_{n+1}}\right| = l$. If $l > 1$, then $(a_n) \to 0$.

Proof:

Let k be any real number such that $1 < k < l$.

Since $\lim \left|\frac{a_n}{a_{n+1}}\right| = l$, there exists $m \in$ N such that

$l-\varepsilon < \left|\frac{a_n}{a_{n+1}}\right| < l+\varepsilon$ for all $n \geq m$.

Choosing $k = l - k$ we obtain $\left|\frac{a_n}{a_{n+1}}\right| > k$ for all $n \geq m$.

Now, fix $n \geq m$. Then

$$\left|\frac{a_n}{a_{m+1}}\right| > k; \left|\frac{a_{m+1}}{a_{m+2}}\right| > k; \ldots \ldots .. + \left|\frac{a_{n-1}}{a_n}\right| > k;$$

Multiplying the above inequalities we get $\left|\frac{a_n}{a_n}\right| > k^{n-m}$.

$\therefore \left|\frac{a_n}{a_n}\right| < k^m \left(\frac{1}{k}\right)^n$.

$\therefore |a_n| < k^m |a_m| \left(\frac{1}{k}\right)^n$

$\therefore |a_n| < Ar^n$ where $A = |a_m| k^m$ is a constant and $r = 1/k$.

Now $k > 1 \Rightarrow 0 < r < 1$.

$\therefore (r^n) \to 0$ (by solved problem 7 of 1.1)

$\therefore (a_n) \to 0$.

Note:

The above theorem is true even if $I = x$,

Theorem 5:

Let (a_*) be any sequence of positive terms and $\lim_{x \to +} \left(\frac{a_n}{a_{n+1}}\right) = l$. If $l < 1$ then $(a_n) \to x$.

Proof:

Proof is similar to that of theorem 4.

Theorem 6:

If the sequences (a_n) and (b_n) converge to 0 and $(b_n)_{\text{i}}$ strictly monotonic decreasing then $\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = \lim_{n \to \infty} \left(\frac{a_n - a_{n+1}}{b_n - b_{n+1}}\right)$ provided the limit on the right hand side exists whether finite or infinite.

Proof:

Case (i) Let $\lim_{n \to \infty} \left(\frac{a_n - a_{n+1}}{b_n - b_{n+1}}\right) = l$, finite.

Let $\varepsilon > 0$ be given. Then there exists $m \in$ N such that

$$l-\varepsilon<\frac{a_n-a_{n+1}}{b_n-b_{n+1}}<l+\varepsilon \text{ for all } n\geq m.$$

Since $b_n-b_{n+1}>0$, we get

$(b_n-b_{n+1})(l-\varepsilon)<a_n-a_{n+1}<(b_n-b_{n+1})(l+\varepsilon)$ for all $n\geq m$.

Let $n>p\geq m$.

Then $\left(b_p-b_{p+1}\right)(l-\varepsilon)<a_p-a_{p+1}<\left(b_p-b_{p+1}\right)(l+\varepsilon)$

$\left(b_{p+1}-b_{p+2}\right)(l-\varepsilon)<a_{p+1}-a_{p+2}<\left(b_{p+1}-b_{p+2}\right)(l+\varepsilon)$

$(b_{n-1}-b_n)(l-\varepsilon)<a_{n-1}-a_n<(b_{n-1}-b_n)(l+\varepsilon)$

Adding the above inequalities, we get

$\left(b_p-b_n\right)(l-\varepsilon)<a_p-a_n<\left(b_p-b_n\right)(l+\varepsilon)$

Taking limit as $n\to\infty$, we get

$b_p(l-\varepsilon)<a_p<b_p(l+\varepsilon)(\text{ since }(a_n),(b_n)\to 0)$

$\therefore\ l-\varepsilon<\frac{a_p}{b_p}<l+\varepsilon\left(\text{ since } b_p>0\right)$

$\therefore\left|\frac{a_p}{b_p}-l\right|<\varepsilon$ for all $p\geq m$.

$\therefore \lim\limits_{n\to-\infty}\frac{a_n}{b_n}=l.$

Case (ii) $\lim_{n\to\infty}\left(\frac{a_n-a_{n+1}}{b_n-b_{n+1}}\right)=x.$

Let $k>0$ be any real number.

Then there exists $m\in$ N such that $\frac{a_c-a_{n+1}}{b_n-b_{n+1}}>k$ for all $n\geq m$.

$\therefore a_n-a_{n+1}>(b_n-b_{n+1})k$ for all $n\geq m$.

Let $n>p\geq m$.

Writing the inequalities for $n=p,p+1,\ldots\ldots,n$ and adding we get

$a_p-a_n>k\left(b_p-b_n\right).$

Taking limit as $n\to\infty$, we get $a_p\geq kb_p$

$\therefore \frac{a_p}{b_p} \geq k$ for all $p \geq m$.

$\therefore \left(\frac{a_n}{b_n}\right)$ diverges to x

Problem 1:

Show that $\lim_{n\to\infty} \frac{1}{n}\left(1+\frac{1}{2}+\cdots\ldots\ldots+\frac{1}{n}\right)=0$.

Solution:

Let $a_n = \frac{1}{n}$.

We know that $(a_n) \to 0$. Hence by Cauchy's first limit theorem

we get $\left(\frac{a_1+a_2+\cdots\ldots\ldots+a_n}{n}\right) \to 0$

$\therefore \left(\frac{1}{n}\left(1+\frac{1}{2}+\cdots\ldots+\frac{1}{n}\right)\right) \to 0$

Problem 2:

Show that $\lim n^{1/n} = 1$.

Solution:

Let $a_n = n$.

$\therefore \lim_{x\to -} \frac{a_{n+1}}{a_n} = \lim_{n\to -} \left(\frac{n+1}{n}\right)$

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1$

$\therefore$ By Cauchy's second limit theorem, we get $\lim_{n\to\infty} n^{1/n} = 1$

Problem 3:

Prove that $\frac{1}{n}[(n+1)'(n+2)\ldots\ldots(n+n)]^{1/n} \to 4/e$.

Solution:

Let $a_n = \frac{1}{n}[(n+1)(n+2)\ldots..(n+n)]^{1/n}$

$$= \left[\frac{(n+1)(n+2)\ldots..(n+n)}{n^n}\right]^{1/n}$$

$$= \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots\left(1+\frac{n}{n}\right)\right]^{1/n}$$

Let $b_n = \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots..\left(1+\frac{n}{n}\right)$ so that $a_n = b_n^{1/n}$.

Now, $\frac{b_{n+1}}{b_n} = \frac{\left(1+\frac{1}{n_+ +1}\right)\left(1+\frac{2}{n+1}\right)\ldots\left(1+\frac{n+1}{n+1}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots\left(1+\frac{n}{n}\right)}$.

$$= (2n+1)(2n+2)\frac{n^n}{(n+1)^{n+2}}$$

$$= \frac{2(2n+1)}{n+1}\frac{n^n}{(n+1)^n}$$

$$= 2\left(\frac{2+1/n}{1+1/n}\right)\frac{1}{(1+1/n)^n}$$

$$\therefore \left(\frac{b_{n+1}}{b_n}\right) \to \frac{4}{e}$$

$\therefore$ By theorem 3.24 we get $\left(b_n^{1/n}\right) \to 4/e$.

$\therefore (a_*) \to 4/e$.

Problem 4:

Prove that $\lim_{x\to-\infty} \frac{x^n}{n!} = 0$.

Solution:

Let $v_s = \frac{x^n}{n!}$.

$$\therefore \frac{a_*}{a_v +} = \frac{x^n}{n!}\frac{(n+1)!}{x^{n+1}} = \frac{n+1}{x}.$$

$$\therefore \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \infty$$

$\therefore (a_t) \to 0$ (by theorem 4)

Problem 5:

Show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0.$

Solution:

Let $a_n = \frac{n!}{n^n}$.

$$\therefore \left|\frac{a_n}{a_{n+1}}\right| = \frac{n!}{n^n}\frac{(n+1)^{n+1}}{(n+1)!}$$
$$= \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$
$$\therefore \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$
$$= e \text{ (by problem 3 of 2.1.)}$$
$$> 1.$$

$\therefore (a_n) \to 0.$ (by theorem 4)

Exercises:

1. Evaluate the limits of the following sequences whose $n^{t_1}\text{ter}^{\text{a}}_{\text{ml}/5}$, given below.

   (a) $\frac{1}{n}\left(1+2^{1/2}+3^{1/3}+\cdots..+n^{1/n}\right)$ (b) $\frac{1}{n}\left(1+2+3^{2/3}+\cdots ...+n^{2/n}\right)$

   (c) $\frac{1}{n}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots..+\frac{1}{2n-1}\right)$ (d) n_n^2 (e) $\left(\frac{(2n)!}{(n!)^2}\right)^{1/n}$ (f) $\left(1+\frac{1}{n}\right)^{n+1}$

   (g) $(1+1/n)^{n+5}$ (h) $\frac{(n!)^{1/n}}{n}$ (i) $\frac{[(a+1)(a+2).....(a+n)]^{1/n}}{n}$ where a is a fixed positite real number.

2. Prove $\lim_{n\to n}\left[\frac{2}{1}\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\cdots\left(\frac{n+1}{n}\right)^n\right]^{1/n} = e.$

3. Prove that $\lim_{n\to=} \frac{n}{(n!)^{1/n}} = e.$

4. Prove that $\lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^n = e.$

5. Prove that $\lim_{x\to\infty}\left(1-\frac{1}{n}\right)^{-n} = e$.

6. Prove that $\lim_{n\to=}\frac{1\cdot3\cdot5\ldots\ldots\ldots(2n-1)}{2\cdot4\ldots\ldots\ldots\ldots2n} = 0$.

7. Show that $\lim_{n\to\infty}\frac{n^5}{2^n} = 0$.

2.3. Sub sequences:

Let (a_{n}) be a sequence. Let (n_{k}) be a strictly increasing sequence of natural numbers. Then $(a_{n_{\mathrm{k}}})$ is called a subsequence of (a_{n}).

Note:

The terms of a subsequence occur in the same order in which they occur in the original sequence.

Examples:

1. (a_{2n}) is a subsequence of any sequence $((a_{\mathrm{n}})$. Note that in this example the interval between any two terms of the subsequence is the same,

(i.e.,) $n_1 = 2, n_2 = 4, n_3 = 6,\ldots\ldots\ldots n_k = 2k.$

2. (a_{n^2}) is a subsequence of any sequence (a_{n}) . Hence $a_{n_1} = a_1, a_{n_2} = a_4, a_{n_3} = a_9,$ ……….. Here the interval between two successive terms of the subsequence goes on increasing as k become large. Thus the interval between various terms of a subsequence need not be regular.

3. Any sequence (a_n) is a subsequence of itself.

4. Consider the sequence (a_n) given by 1, 0, 1, 0,…….. Now, (b_n) given by 1,1,1,…. Is a subsequence of (a_n). Here (a_n) is not convergent whereas the subsequence (b_n) converges to 1. Thus a subsequence of non-convergent sequence can be a convergent sequence.

Note: A subsequence of a given subsequence $(a_{n_{\mathrm{k}}})$ of a sequence (a_n) is again a subsequence of (a_n).

Theorem 1:

If a subsequence (a_n) converges to l, then every subsequence (a_{n_k}) of (a_n) also converges to l.

Proof:

Let $\varepsilon > 0$ be given .

Since $(a_n) \to l$ there exists m∈ N such that

$|a_n - l| < \varepsilon$ for all $n \geq m$. ………….(1)

Now choose $n_k \geq m$

Then k≥ $k_0 \Rightarrow n_k \geq n_{k_0}$

$\Rightarrow n_k \geq$ m.

$\Rightarrow |a_{n_k} - l| < \varepsilon\ for\ all$ k≥ k_0.

$\therefore (a_{n_k}) \to l$.

Note:

1. If a subsequence of a sequence convergence, then the original sequence need not converge. (refer example 4)
2. If a sequence (a_n) has two subseuences converging to two limits, then (a_n) does not converge. For example, consider the sequence (a_n) given by

$$a_n = \begin{cases} 1/\text{n} & \text{if } n \text{ is even} \\ 1 + \frac{1}{n} & \text{if } n \text{ is odd} \end{cases}.$$

Here the subsequence $(a_{2n}) \to 0$ and the subsequence $(a_{2n-1}) \to 1$. Hence the given sequence (a_n) does not converge.

Theorem 2:

If the sub sequences (a_{2n-1}) and (a_{2n}) of a sequence (a_n) converge to the same limit l then (a_n) also converges to l.

Proof:

Let $\varepsilon > 0$ be given. Since $(a_{2n-1}) \to l$ there exists $n_1 \in$ N such that $|a_{2n-1} - l| < \varepsilon$ for all $2n - 1 \geq n_1$.

Similarly there exists $n_2 \in$ N such that $|a_{2n} - l| < \varepsilon$ for all $2n \geq n_1$.

Let $m = \max\{n_1, n_2\}$.
Clearly $|a_n - l| < \varepsilon$ for all $n \geq m$.
$\therefore (a_n) \to l$.

Note:

The above result is true even if we have $l = \infty$ or $-\infty$.

Definition:

Let (a_n) be a sequence. A natural number m is called a peak point of the sequence (a_n) if $a_n < a_m$ for all $n > m$.

Example:

1. For the sequence $(1/n)$, every natural number is a peak point and hence the sequence has infinite number of peak points. In general, for a strictly monotonic decreasing sequence every natural number is a peak point.
2. Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, -1, -1, \ldots$ ..... Here 1,2,3 are the peak points of the sequence.
3. The sequence $1,2,3, \ldots \ldots$ has no peak point. In general, a monotonic increasing sequence bas no peak point.

Theorem 3:

Every sequence (a_n) has a monotonic subsequence.

Proof:

Case (i):

(a_n) has infinite number of peak points.

Let the peak points be $n_1 < n_2 < \cdots .. < n_k < \cdots ...$

Then $a_{n_1} > a_{n_2} > \cdots .. > a_{n_4} > \cdots ..$

$\therefore$ (a_n) is a monotonic decreasing subsequence of (a_n).

Case (ii):

(a_n) has only a finite number of peak points or no peak poin,

Choose a natural number n_1 such that there is no peak point grom point of (a_n), there exists $n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$. Again since n_2 is not a peak point, there exists $n_3 > n_2$ such that $a_{n_3} \geq a_{n_2}$. Repeating this process we get a monotonic incteatis, subscquenc (a_{n_k}) of (a_n).

Theorem 4:

Every bounded sequence has a convergent subsequence.

Proof:

Let (a_n) be a bounded sequence. Let (a_{nn}) be a monotonie subseque qus of (a_n).

Since (a_n) is bounded (a_n) is also bounded:

$\therefore$ (a_{n_k}) is a bounded monotonic sequence and hence convergent,

$\therefore$ (a_{n_k}) is a convergent subsequence of (a_n).

Exercises:

1. Prove that if a sequence (a_a) diverges to ∞ then every subsequertes of (a_n) also diverges to ∞.
2. Prove that if a sequence (a_n) diverges to $-\infty$ then every subsequence of (a_n) also diverges to $-\infty$.
3. Give examples of

   (i) a sequence which does not diverge to ∞ bal has a subsequence diverging to ∞ (ii) a sequence which does not diverge to $-\infty$ but has a subsequence diverging to $-\infty$.

(iii) a sequence (a_n) bavith two subsequences, one diverging to ∞ and the other diverging $-\infty$,

4. Prove, that each of the following sequences is not convergent by exhibiting two sub sequences converging to two different limits.

   (i) $1, \frac{1}{2} 1, \frac{1}{3}, 1, \frac{1}{4}, \ldots \ldots .. ,1, \frac{1}{n}, \ldots \ldots$

   (ii) $1,2,1,3,1,4, \ldots \ldots$

   (iii) $((-1)^n)$.

2.4. Limit Points:

Definition:

Let (a_n) be a sequence of real numbers a is called a limit point or a cluster point of the sequence (a_n) if given $\varepsilon > 0$, there exists infinite number of terms of the sequence in $(a - \varepsilon, a + \varepsilon)$. If the sequene (a_n) is not bounded above then x is a limit point of the sequence. If (a_n) is not bounded below then $-\infty$ is a limit point of the sequence.

Examples:

1. Consider the sequence $1,0,1,0, \ldots$. For this sequence 1 is a limit point since given $\varepsilon > 0$, the interval $(1 - \varepsilon, 1 + \varepsilon)$ contains infinitely many terms $a_1, a_3, a_5, \ldots$ of this sequence. Similarly, 0 is also a limit point of this sequence.

2. If a sequence (a_n) converges to l then l is a point of the sequence. For, given $\varepsilon > 0$, there cxists $m \in N$ such that $a_n \in (l - \varepsilon, l + \varepsilon)$ for all $n \geq m$.
   $\therefore (l - \varepsilon, l + \varepsilon)$ contains infinitcly many terms of the sequence.

3. The sequence $(a_n) = 1,2,3, \ldots . n \ldots$ is not bounded above and hence ∞ is a limit point.

4. The sequence $(a_n) = 1, -1, 2, -2, \ldots \ldots n, -n \ldots \ldots$ is neither bounded above nor bounded below. Hence ∞ and $-\infty$ are limit points of the

Theorem 1:

Let (a_n) be a sequence. A real number a is a limit point of (a_n) iff there exists a subsequence (a_{n_k}) of (a_n) converging to a.

Proof:

Suppose there exists a subsequence (a_{n_k}) of (a_n) converging to a.

Let $\varepsilon > 0$ be given. Then there cxists $k_0 \in N$ such that $a_n \in (a-\varepsilon, a+\varepsilon)$ for all $k \geq k_0$.

$\therefore$ $(a-\varepsilon, a+\varepsilon)$ contains infinitely many terms of the sequence (a_n).

$\therefore$ a is a limit point of the sequence (a_n).

Conversely suppose a is a limit point of (a_n).

Then for each $\varepsilon > 0$ the interval $(a-\varepsilon, a+\varepsilon)$ contaits infinitely many terms of the sequence.

In particular we can find $n_1 \in$ N such that $(a_{n_k}) \in (a-1, a+1)$.

Also we can find $n_2 > n_1$ such that $a_{n_2} \in \left(a-\frac{1}{2}, a+\frac{1}{2}\right)$.

Proceeding like this we can find natural numbers $n_1 < n_2 < n_3$ ... . such that

$a_{n_k} \in (a-1/k, a+1/k)$.

Clearly (a_n) is a subsequence of (a_n) and $|a_{n_k} - a| < 1/k$

For any $\varepsilon > 0, |a_{n_k} - a| < \varepsilon$ if $k > 1/\varepsilon$.

$\therefore (a_{n_k}) \to a$.

Theorem 2:

Every bounded sequence has at least one limit point.

Proof:

Let (a_n) be a bounded sequence. Then there exists a convergent subsequence (a_{n_k}) of (a_n) converging to l (say) (by theorem 2 of 1.2).

Hence l is a limit point of (a_n).

Note:

In general every sequence (a_n) has at least one limit point (finite or intinitc).

Theorem 3:

A sequence (a_n) converges to l iff (a_n) is bounded and l is the only limit point of the sequence.

Proof:

Let $(a_n) \to l$. Then (a_n) is bounded (by theorem 2 of 1.2).

Also l is a limit point of the sequence (a_n) (by example 2 of 2.4).

Now suppose l_1 is any other limit point of (a_n). Then there exist a subsequence (a_{n_k}) of (a_n) such that $(a_n) \to l_1$.

Conversely, suppose l is the only limit point of (a_n). Suppose (a_n) does not converge to l. Then there exists at least one $\varepsilon > 0$ such that infinitely many terms of the sequence lie outside $(l - \varepsilon, l + \varepsilon)$. Hence we can find a subsequence (a_{n_k}) of (a_n)

such that $a_{n_k} \notin (l - \varepsilon, l + \varepsilon)$ for all k.

Since (a_n) is a bounded sequence, (a_{n_k}) is also a bounded sequence. Hence (a_{n_k}) has also a limit point by theorem 2, say, l' and $l' \neq l$.

$\therefore (a_n)$ has two limit points l *and* l which is a contradiction. Hence $(a_n) \to l$.

Exercises:

1. Find all the limit points of each of the following sequences.

   i)(1/n) ii) (n^2) iii) $((-1)^n)$ iv) (2n-1)

2. Construct a sequence having exactly 10 limit points.

2.5. Cauchy Sequences:

In this section we prove a necessary and sufficient condition for given sequence to be convergent. This criterion involves only the terms of sequence under consideration and hence can be used to test the converge of a sequence without having any idea of its limit.

Definition:

A sequence (a_n) is said to be a Cauchy sequence if given $\varepsilon > 0$, there exists $n_0 \in N$ such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq n_0$.

Note:

In the above definition the condition $|a_n - a_m| < \varepsilon$ for all $n, m \geq n_0$ can be written in the following equivalent form, naze $|a_{n+p} - a_n| < \varepsilon$ for all $n \geq n_0$ and for all positive integers p.

Example 1:

The sequence $(1/n)$ is a Cauchy sequence.

Proof:

Let $(a_n) = (1/n)$. Let $\varepsilon > 0$ be given.

Now, $|a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right|$.

$\therefore$ If we choose n_0 to be any positive integer greater than $\frac{1}{\varepsilon}$,

we get $|a_n - a_m| < \varepsilon$ for all $n, m \geq n_0$.

$\therefore$ $(1/n)$ is a Cauchy sequence.

Example 2:

The sequence $((-1)^n)$ is not a Cauchy sequence.

Proof:

Let $(a_n) = ((-1)^n)$.

$\therefore |a_n - a_{n+1}| = 2$

$\therefore$ If $\varepsilon < 2$, we cannot find n_0 such that $|a_n - a_{n+1}| < \varepsilon$ for all $n \geq n_0$.

$\therefore$ $((-1)^n)$ is not a Cauchy sequence.

Example 3:

(n) is not a Cauchy sequence.

Proof:

Let $(a_n) = (n)$.

$\therefore |a_n - a_m| \geq 1$ if $n \neq m$.

$\therefore$ If we choose $\varepsilon < 1$,

we cannot find n_0 such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq n_0$.

$\therefore (n)$ is not a Cauchy sequence.

Theorem 1:

Any convergent sequence is a Cauchy sequence.

Proof:

Let $(a_n) \to l$. Then given $\varepsilon > 0$, there exists $\dot{n}_0 \in$ N

such that $|a_n - l| < \frac{1}{2}\varepsilon$ for all $n \geq n_0$.

$$\begin{aligned}\therefore |a_n - a_m| &= |a_n - l + l - a_m| \\ &\leq |a_n - l| + |l - a_m| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \text{ for all } n, m \geq n_0.\end{aligned}$$

$\therefore (a_n)$ is a Cauchy sequence.

Theorem 2:

Any Cauchy sequence is a bounded sequence.

Proof:

Let (a_n) be a Cauchy sequence.

Let $\varepsilon > 0$ be given. Then there exists $n_0 \in N$

such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq n_0$.

$\therefore |a_n| < |a_{n_0}| + \varepsilon$ for $n \geq n_0$.

Now, let $k = \max\{|a_1|, |a_2|, \ldots\ldots, |a_{n_e}| + \varepsilon\}$.

Then $|a_n| \leq k$ for all n. Hence (a_n) is a bounded sequence.

Theorem 3:

Lel(a_n) be a Cauchy seguence. If (a_n) has a subsequence (a_{n_k}) converging to l, then $(a_n) \to 1$.

Proof:

Let $\varepsilon > 0$ be given. Then there exists $n_0 \in N$ such that

$|a_n - a_m| < \frac{1}{2}\varepsilon$ for all $n, m \geq n_0$ ....(1)

Also since $(a_{n_k}) \to l$, there exists $k_0 \in$ N such that $\left|a_{n_1} - l\right| < \frac{1}{2}\varepsilon$ for all $k \geq k_0$ ……….(2)

Choose n_k such that $n_k \geq n_k$ and n_0.

Then $|a_n - l\,| = \left|a_n - a_{n_k} + a_{n_k} - l\right|$

$$\leq \left|a_n - a_{n_k}\right| + \left|a_{n_k} - l\right|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \text{ for all } n \geq n_0.$$

Hence $(a_n) \to l$.

Note:

In theorem 1 we proved that any convergent sequence is a Cauchy sequence. We now proceed to prove that the converse of the above theorem is also true. That is, any Cauchy sequence in R is convergent. This is known as the Canchy's general principle of convergence and this property of the real number system is known as the completeness of R and we say that R is complete.

Theorem 4: (Cauchy's general principle of convergence)

A sequence (a_n) in **R** is convergent iff it is a Cauchy sequence.

Proof:

In theorem 1 we have proved that any convergent sequence is a Cauchy sequence.

Conversely, let (a_n) be a Cauchy sequence in **R**.

∴ (a_n) is a bounded sequence (by theorem 2).

∴ There exists a subsequence (a_n) of (a_n) such that $(a_n) \to l$

∴ $(a_n) \to l$ (by theorem 3).

Note:

There are Cauchy sequences in Q which are not convergent in Q . For example, the sequence1 ,1.4, 1.41, 1.414 … , ……. whose terms are successive decimal expressions of $\sqrt{2}$ is a Cauchy sequence in Q which is not convergent in Q .

Exercises:

1. Show that the following are Cauchy sequences.

   (a) $\left(\frac{1}{n^2}\right)$ (b) $\left(1 + \frac{1}{n}\right)$

   (c) $\left(\frac{(-1)^n}{n}\right)$ (d) $\left(\frac{1}{n!}\right)$

2. Show that the following are not Cauchy sequences.

   (a) $\left((-1)^n + \frac{1}{n}\right)$ (b) $((-1)^n n)$ (c) (n^2)

Unit III

Series of positive terms: Infinite series – Comparison test.

Chapter 3: Sections 3.1, 3.2

Series of Positive Terms:

3.1. Infinite Series:

Definition:

Let $(a_n) = a_1, a_2, \ldots\ldots, a_n, \ldots..$ be a sequence of real number. Then the formal expression $a_1 + a_2 + \cdots. + a_n + \cdots..$ is called an infinite series of real numbers and is denoted by $\sum_1^\infty a_n$ or $\sum a_n$.

Let $s_1 = a_1; s_2 = a_1 + a_2; s_3 = a_1 + a_2 + a_3;$
$s_n = a_1 + a_2 + \cdots.. + a_n.$

Then (s_n) is called the sequence of partial sums of the given series Σa_n

The series Σa_n is said to converge, diverge or oscillate accos as the sequence of partial sums (s_n) converges, diverges or oscillates.

If $(s_n) \to s$, we say that the series Σa_n converges to the sum s.

We note that the behaviour of a series does not change if a fil number of terms are added or altered.

Example 1:

Consider the series $1 + 1 + 1 + 1 + \cdots \ldots \ldots$

Here $s_n = n$. Clearly the sequence (s_n) diverges to ∞. Hence the given series diverges to ∞.

Example 2:

Consider the geometric series $1 + r + r^2 + +r^n +$

Here, $s_n = 1 + r + r^2 + \cdots. + r^{n-1} = \frac{1-r^n}{1-r}$.

Case (i) $0 \le r < 1$.

Then $(r^n) \to 0$ (refer problem 7 of 1.7)

$\therefore (s_n) \to \frac{1}{1-r}$.

$\therefore$ The given series converges to the sum $1/(1-r)$.

Case (ii) $r > 1$.

Then $s_n = \frac{r^n-1}{r-1}$.

Also $(r^n) \to \infty$ when $r > 1$.

Hence the series diverges to ∞.

Case (iii) $r = 1$.

Then the series becomes $1 + 1 + \cdots$.

$\therefore (s_n) = (n)$ which diverges to ∞.

Case (iv) $r = -1$.

Then the series becomes $1 - 1 + 1 - 1 + \cdots\ldots$.

$\therefore s_n = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd} \end{cases}$.

$\therefore (s_n)$ oscillates finitely.

Hence the given series oscillates finitely.

Case (v) $r < -1$.

$\therefore (r^n)$ oscillates infinitely (by problem 7 of 1.7).

$\therefore (s_n)$ oscillates infinitely.

Hence the given series oscillates infinitely.

Example 3:

Consider the series $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots \ldots\ldots + \frac{1}{n!} +$

Then $s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots \ldots + \frac{1}{(n-1)!}$.

The sequence $(s_n) \to e$ (refer problem 1 of 2.1).

$\therefore$ The given series converges to the sum e.

Example 4:

Consider the series $1 + \frac{1}{2} + \frac{1}{3} + + \frac{1}{n} +$

Then $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

Here $(s_n) \to \infty$ (refer solved problem 5 of 2.1).

$\therefore$ The given series diverges to ∞.

Note 1:

Let $\sum a_n$ be a series of positive terms. Then (s_n) is a monotonic increasing sequence. Hence (s_n) converges or diverges to ∞ according as (s_n) is bounded or unbounded. Hence the series Σa_n converges or diverges to ∞. Thus a series of positive terms cannot oscillate

Note 2:

Let $\sum a_n$ be a convergent series of positive terms converging to the sum s. Then s is the l.u.b of (s_n). Hence $s_n \leq s$ for all n.

Also given $\varepsilon > 0$ there exists $m \in N$ such that $s - \dot{\varepsilon} < s_n$

Hence $s - \varepsilon < s_n \leq s$ for all $n \geq m$.

Theorem 1:

Let Σa_n be a convergent series converging to the sum s. Then $\lim_{n\to\infty} a_n = 0$.

Proof:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1})$$
$$= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}.$$
$$= s - s = 0$$

Note 1:

The converse of the above theorem is not true.

(i.e.) If $\lim a_n = 0$, then $\sum a_n$ need not converge, For example, consider the series $\sum \frac{1}{n}$.Here

$\lim_{n\to\infty} \frac{1}{n} = 0$. However the series $\sum \frac{1}{n}$ diverges. (By example 4 of 3.1)

Note 2:

If $\lim a_n \neq 0$ then the series $\sum a_n$ is not convergent. If further $\sum a_n$ is a series of positive terms then the series cannot oscillate and hence the series diverges.

Theorem 2:

Let Σa_n converge to a and Σb_n converge to b. Then $\Sigma(a_n \pm b_n)$ converges to $a \pm b$ and $\Sigma k a_n$ converges to ka.

Proof:

Let $s_n = a_1 + a_2 + \cdots .. + a_n$ and

$t_n = b_1 + b_2 + \cdots .. + b_n$.

Then $(s_n) \to a$ and $(t_n) \to b$.

$\therefore (s_n \pm t_n) \to a \pm b$ (refer theorem 3.8)

Also $(s_n \pm t_n)$ is the sequence of partial sums of $\Sigma(a_n \pm b_n)$.

$\therefore \Sigma(a_n \pm b_n)$ converges to $a \pm b$.

Similarly $\sum k a_n$ converges to ka.

Theorem 3: (Cauchy's general principle of convergence)

The series Σa_n is convergent iff given $\varepsilon > 0$ there exists $n_0 \in$ **N** II such that $|a_{n+1} + a_{n+2} + \cdots \ldots . + a_{n+p}| < \varepsilon$ for all $n \geq n_0$ and for all positive integers p.

Proof:

Let $\sum a_n$ be a convergent series.

Let $s_n = a_1 + \cdots \ldots .+a_n$.

∴ (s_n) is a convergent sequence.

∴ (s_n) is Cauchy sequence (by theorem 1 of 3.1).

∴ There exists $n_0 \in$ N such that $|s_{n+p} - s_n| < \varepsilon$ for all $n \geq n_0$ and for all $p \in \mathbf{N}$.

∴ $|a_{n+1} + a_{n+2} + \cdots \ldots + a_{n+p}| < \varepsilon$ for all $n \geq n_0$ and for all $p \in N$.

Conversely if $|a_{n+1} + a_{n+2} + \cdots \ldots + a_{n+p}| < \varepsilon$ for an, and for all $p \in$ N then (s_n) is a Cauchy sequence in R and hence, convergent. (by theorem 4 of 3.1).

∴ The given series converges.

Solved Problems.

Problem 1:

Apply Cauchy's general principle of convergence to show the series $\Sigma(1/n)$ is not convergent.

Solution:

Let $s_n = 1 + \frac{1}{2} + \cdots \ldots + \frac{1}{n}$.

Suppose the series $\Sigma(1/n)$ is convergent.

∴ By Cauchy's general principle of convergence, given: there exists $m \in$ N such that $|s_{n+p} - s_n| < \varepsilon$ for all $n \geq m$ and $f_0 p \in N$.

∴ $\left|\left(1 + \frac{1}{2} + \cdots \ldots + \frac{1}{n+p}\right) - \left(1 + \frac{1}{2} + \cdots \ldots + \frac{1}{n}\right)\right| < \varepsilon$ for all $n \geq n_0$and for all $p \in$ N.

∴ $\left|\frac{1}{n+1} + \frac{1}{n+2} + \cdots . + \frac{1}{n+p}\right| < \varepsilon$ for all $n \geq m$ and for all $p \in \mathbf{N}$.

In particular if we take $n = m$ and $p = m$

we obtain $\frac{1}{m+1} + \frac{1}{m+2} + \cdots . + \frac{1}{m+m} > \frac{1}{2m} + \cdots + \frac{1}{2m} = \frac{1}{2}$.

∴ $\frac{1}{2} < \varepsilon$ which is a contradiction since $\varepsilon > 0$ is arbitrary.

∴ The given series is not convergent.

Problem 2:

Applying Cauchy's general principle of convergence provel $1 - \frac{1}{2} + \frac{1}{3} - \cdots \ldots . + (-1)^n \frac{1}{n} + \cdots$.. is convergent.

Solution:

Let $s_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots \ldots + \frac{(-1)^n}{n}$.

$$\therefore |s_{n+p} - s_n| = \left|\frac{1}{n+1} - \frac{1}{n+2} + \cdots \ldots + \frac{(-1)^{p-1}}{n+p}\right|$$

Now, $\frac{1}{n+1} - \frac{1}{n+2} + \cdots .. + \frac{(-1)^{p-1}}{n+p}$

$$= \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \begin{cases} \frac{1}{n+p-1} - \frac{1}{n+p} & \text{if } p \text{ is even} \\ \frac{1}{n+1} \text{ if } p \text{ is odd} \end{cases}$$

> 0 .

$$\therefore |s_{n+p} - s_n| = \frac{1}{n+1} - \frac{1}{n+2} + \cdots . + \frac{(-1)^{p-1}}{n+p}$$

$$= \frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \cdots \ldots$$

$$< \frac{1}{n+1}$$

$$< \varepsilon \text{ provided } n > \left(\frac{1}{\varepsilon} - 1\right)$$

∴ By Cauchy's general principle of convergence, the given series is convergent.

Exercises:

1. Show that the series $\sum\left(\frac{1}{2^n}\right)$ converges to the sum 1 .
2. Show that the series $1 + 2 + 3 +$ diverges to ∞.
3. Show that if Σa_n converges and Σb_n diverges then $\Sigma(a_n + b_n)$ diverges.
4. Prove that if Σc_n is a convergent series of positive terms then so is $\sum a_n c_n$ where (a_n) is a bounded sequence of positive terms.
5. Prove that if Σd_n is a divergent sequence of positive Iermis the is $\Sigma a_n d_n$ where (a_n) is a sequence with a positive lower bound.
6. Show that $\frac{2}{5} + \frac{4}{5^2} + \frac{2}{5^3} + \frac{4}{5^4} + \frac{2}{5^5} + \frac{4}{5^6} + \cdots \ldots = \frac{7}{12}$

   (Hint : Express this serics as the sum of two geometric series),
7. Prove that a sequence (a_n) is convergent iff $\sum(a_{n+1} - a_n)$ in convergent.
8. Let a and b be two positive real numbers. Show that the series $a + b + a^2 + b^2 + a^3 + b^3 + \cdots \ldots$.. converges if both a and $b < 1$ and dive. ges if either $a \geq 1$ or $b \geq 1$.

3.2. Comparison Test:

In the next few sections we develop some standard tests for convergence of series of positive terms. For the rest of this chapter we confine ourselves to series of positive terms.

Theorem 1: (Comparison Test)

(i) Let $\sum c_n$ be a convergent series of positive terms. Let $\sum a_n$ be another series of positive terms. If there exists $m \in \mathrm{N}$ such that $a_n \leq c_n$ for all $n \geq m$ then $\sum a_n$ is also convergent. (ii) Let $\sum d_n$ be a divergent series of positive terms Let $\sum a_n$ be another series of positive terms. If there exists $m \in \mathrm{N}$ such that $a_n \geq d_n$ for all $n \geq m$ then $\sum a_n$ is also divergent.

Proof:

(i) Since the convergence or divergence of a series is not altered by the removal of a finite number of terms we may assume without loss of generality that $a_n \leq c_n$. for all n.

Let $S_n = c_1 + c_2 + \cdots .. + c_{n.}$ and $t_n = a_1 + a_2 + \cdots .. + a_n$.

Since $a_n \leq c_n$ we have $t_n \leq s_n$.

Now, since $\sum c_n$ is convergent, (s_n) is a convergent sequence.

∴ (s_n) is a bounded sequence. (by theorem 2 of sec 1.1)

∴ There exists a real positive number k such that $s_n \leq k$ for all n.

∴ $t_n \leq k$ for all n

Hence (t_n) is bounded above.

Also (t_n) is a monotonic increasing sequence.

∴ (t_n) converges (by theorem 1 of 2.1).

∴ $\sum a_n$ converges.

(ii) Let $\sum d_n$ diverge and $a_n \geq d_n$ for all n.

∴ $t_n \geq S_n$

Now, (s_n) diverges to ∞.

∴ (s_n) is not bounded above.

∴ (t_n) is not bounded above.

Further (t_n) is monotonic increasing and hence (t_n) diverges to ∞.

∴ Σa_n diverges to ∞.

Theorem 2:

(i) If Σc_n converges and if $\lim_{n\to\infty} \left(\frac{a_n}{c_n}\right)$ exists and is finite then Σa_n also converges.

(ii) If $\sum d_n$ diverges and if $\lim_{n\to=} \left(\frac{a_n}{d_n}\right)$ exists and is greater than zero then Σa_n diverges.

Proof:

(i) Let $\lim_{n\to\infty}\left(\frac{a_n}{c_n}\right) = k$.

Let $\varepsilon > 0$ be given. Then there exists $n_1 \in$ N such that

$\frac{a_n}{c_n} < k + \varepsilon$ for all $n \geq n_1$.

$\therefore a_n < (k+\varepsilon)c_n$ for all $n \geq n_1$.

Also since Σc_n is a convergent series, $\Sigma(k+\varepsilon)c_n$ is also a convergent series.

$\therefore$ By comparison test Σa_n is convergem.

(ii) Let $\lim\limits_{n\to-\infty}\left(\frac{a_n}{d_n}\right) = k > 0$.

Choose $= \frac{1}{2}k$: Then there exists $n_1 \in N$ such that

$k - \frac{1}{2}k < \frac{a_n}{d_n} < k + \frac{1}{2}k$ for all $n \geq n_1$.

$$\therefore \frac{a_n}{d_n} > \frac{1}{2}k \text{ for all } n \geq n_1.$$

$$\therefore a_n > \frac{1}{2}kd_n \text{ for all } n \geq n_1.$$

Since Σd_n is a divergent series, $\Sigma\frac{1}{2}kd_n$ is also divergent series.

$\therefore$ By comparison test, $\sum a_n$ diverges.

Theorem 3:

(i) Let $\sum c_n$ be a convergent series of positive terms. Let $\sum a_n$ be another series of positive terms. If there exists $m \in N$ such that $\frac{q_{n+1}}{a_n} \leq \frac{c_n+1}{c_n}$ for all $n \geq m$, then Σa_n is convergent.

(ii) Let $\sum d_n$ be a divergent series of positive terms. Let $\sum a_n$ be another series of positive terms. If there exists $m \in$ N such that $\frac{a_{n+1}}{a_n} \geq \frac{d_{n+1}}{d_n}$ for all $n \geq m$, then $\sum a_n$ is divergent.

Proof:

(i) $\frac{a_{n+1}}{c_n+1} \leq \frac{a_n}{c_n}$ (since $\frac{a_{n+1}}{a_n} \leq \frac{c_{n+1}}{c_n}$)

$$\therefore \left(\frac{a_n}{c_n}\right) \text{ is a monotonic decreasing sequence.}$$

$$\therefore \frac{a_n}{c_n} \leq k \text{ for all } n \text{ wnere } k = \frac{a_1}{c_1}.$$

$$\therefore a_n \leq kc_n \text{ for all } n \in \text{N}.$$

Now, Σc_n is convergent. Hence Σkc_n is also a convergent series of positive terms.

$\therefore \Sigma a_n$ is also convergent (by theorem 1).

(ii) Proof is similar to that of (i).

Note:

1. Theorems 2 and 3 are alternative, forms of the comparison test mentioned in theorem 1 and these forms of the comparison test are often easier to work with.
2. The comparison test can be used only if we already have a large number of series whose convergence or divergence are known. We know that a geometric series Σr^n converges if $0 \leq r < 1$ and diverges if $r \geq 1$. In the following theorem we give another family of series whose behaviour is known.

Theorem 4:

The harmonic series $\sum \frac{1}{n^p}$ converges if $p > 1$ and divergence if $p \leq 1$.

Proof:

Case (i)

Let $p = 1$. Then the series becomes $\Sigma(1/n)$ which divetron (refer example 4 of 3.2).

Case (ii)

Let $p < 1$. Then $n^p < n$ for all n.

$$\therefore \frac{1}{n^p} > \frac{1}{n} \text{ for all } n.$$

$$\therefore \text{ By comparison test } \sum \frac{1}{n^p} \text{ diverges.}$$

Case (iii) Let $p > 1$.

$$\text{Let } s_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \dots \dots + \frac{1}{n^p}$$

$$s_{2^{n+1}} = 1 = 1 + \frac{1}{2^p} + \cdots \dots \dots + \frac{1}{(2^{n+1} - 1)^p}$$

$$= 1 + \left(\frac{1}{2^p}\frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \cdots ..$$

$$\dots \dots \dots + \left(\frac{1}{(2^n)^p} + \frac{1}{(2^{n+1}y^p} + \cdots .. + \frac{1}{(2^{n+1} - 1)^p}\right)$$

$$< 1 + 2\left(\frac{1}{2^p}\right) + 4\left(\frac{1}{4^p}\right) + \cdots \dots \dots + 2^n\left(\frac{1}{(2^n)^p}\right)$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{p-2}} + \cdots \dots \dots + \frac{1}{2^{(p-1)}}$$

$$\therefore s_{2^{n+1}} - 1 < 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \cdots .. + \left(\frac{1}{2^{p-1}}\right)^n$$

Now, since $p > 1, p - 1 > 0$. Hence $\frac{1}{2^{p-1}} < 1$.

$$\therefore 1 + \left(\frac{1}{2^{p-1}}\right) + \left(\frac{1}{2^{p-1}}\right)^2 + \cdots \ldots \ldots + \left(\frac{1}{2^{p-1}}\right)^n$$

$$< \frac{1}{1 - \frac{1}{2^{p-1}}} = k(\text{ say })$$

$$\therefore s_2^{n+1} - 1 < k.$$

Now let n be any positive integer. Choose $m \in$ N such that $n \leq 2^{m+1} - 1$.

Since (s_n) is a monotonic increasing sequence, $s_n \leq s_2^{m+1}{}_{-1}$.

Hence $s_n < k$ for all n.

Thus (s_n) is a monotonic increasing sequence and is bounded above.

$\therefore$ (s_n) is convergent.

$\therefore$ $\sum \frac{1}{n^p}$ is convergent.

Problem 1:

Discuss the convergence of the series $\sum \frac{1}{\sqrt{(n^3+1)}}$

Solution:

$\frac{1}{\sqrt{(n^3+1)}} < \frac{1}{n^{3/2}}$.

Also $\sum_{n^{3/2}}$ is convergent (by theorem 4).

$\therefore$ By comparison test, $\sum \frac{1}{\sqrt{(n^3+1)}}$ is convergent.

Problem 2:

Discuss the convergence of the series $\sum \frac{\sqrt{(n+1)} - \sqrt{n}}{n^p}$

Solution:

$$a_n = \frac{\sqrt{(n+1)} - \sqrt{n}}{n^p}$$

$$= \frac{n+1-n}{n^p(\sqrt{(n+1)}+\sqrt{n})}$$

$$= \frac{1}{n^p(\sqrt{(n+1)}+\sqrt{n})}$$

Now, let $b_n = \frac{1}{n^{p+1/2}}$.

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to-\infty} \frac{n^{n^{p+1/2}}}{n^p(\sqrt{(n+1)}+\sqrt{n})}$$

$$= \lim_{n\to-\infty} \frac{1}{\sqrt{(1+1/n)}+1)}$$

$$= \frac{1}{2}.$$

Also $\sum b_n$ is convergent if $p + \frac{1}{2} > 1$ and divergent if $p + \frac{1}{2} \leq 1$ (refer theorem 4).

$\therefore \Sigma a_n$ is convergent if $p > \frac{1}{2}$ and divergent if $p \leq \frac{1}{2}$.

Problem 3:

Discuss the convergence of the series $\sum \frac{1^2+2^2+\cdots\ldots+n^2}{n^4+1}$

Solution:

Let $a_n = \frac{1^2+2^2+\cdots\ldots+n^2}{n^4+1}$.

$$= \frac{n(n+1)(2n+1)}{6(n^4+1)}$$

Now, let $b_n = \frac{1}{n}$.

$$\therefore \lim_{\cdots\to} \frac{a_n}{b_n} = \lim_{\cdots\infty} \frac{n^2(n+1)(2n+1)}{6(n^4+1)}$$

$$= \lim_{\cdots\infty} \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6\left(1+\frac{1}{n^4}\right)}$$

$$= \frac{1}{3}.$$

Also Σb_n is divergent (by theorem 4).

$\therefore \Sigma a_n$ is divergent (by theorem 2)

Problem 4:

Discuss the convergence of the series $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \cdots$ ....

Solution:

Let $a_n = \frac{n^n}{(n+1)^{n+1}}$

Let $b_n = \frac{1}{n}$.

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{n+1}}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}}$$

$$= \frac{1}{e} > 0$$

Also Σb_n is divergent.

$\therefore$ Σa_n is divergent (by theorem 2).

Problem 5:

Discuss the convergence of the series$\sum_3^\infty (\log\log n)^{-\log n}$.

Solution:

Let $a_n = (\log\log n)^{-\log n}$

$\therefore$ $a_n = n^{\theta_n}$ where $\theta_n = \log(\log\log n)$.

Since $\lim_{n\to\infty} \log\log\log n = \infty$ there exists $m \in \mathrm{N}$

Such that $\theta_n \geq 2$ for all $n \geq m$.

$\therefore n^{\theta_n} \leq n^{-2}$ for all $n \geq m$

$\therefore a_n \leq n^{-2}$ for all $n \geq m$

Also $\sum n^{-2}$ is convergent.

$\therefore$ By comparison test the given series is convergent.

Problem 6:

Show that $\sum \frac{1}{4n^2-1} = \frac{1}{2}$.

Solution:

Let $a_n = \frac{1}{4n^2-1}$.

Clearly, $a_n < \frac{1}{n^2}$.

Also $\sum \frac{1}{n^2}$ is convergent (by theorem 4)

$\therefore$ By comparison test, the given series coverges.

Now, $a_n = \frac{1}{4n^2-1} = \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$ (by partial fractions)

$$\therefore s_n = a_1 + a_2 + \cdots . + a_n$$
$$= \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots\ldots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$$
$$= \frac{1}{2}\left[1-\frac{1}{2n+1}\right]$$
$$\therefore \lim_{n\to\infty} s_n = \frac{1}{2}.$$
$$\therefore \sum \frac{1}{4n^2-1} = \frac{1}{2}.$$

Exercises:

1.Discuss the convergence of the following series whose n^{th} terms are given below.

(i) $\frac{5+n}{3+n^2}$ (ii) $\frac{2n}{n^2+1}$ (iii) $\frac{\sqrt{n}}{n^2-1}$ (iv) $\frac{n^4-5n^2+1}{n^6+3n^2+2}$ (v) $\frac{1}{n\sqrt{(n^2+1)}}$ (vi) $\frac{n}{(n^2+1)^{2/3}}$

(vii) $\frac{n}{(n^2+1)^{3/2}}$ (viii) $\frac{1}{n-\sqrt{n}}$ (ix) $\frac{n(n+1)}{(n+2)(n+3)(n+4)}$ (x) $\frac{1}{a+nx}$ (xi) $\frac{(n+1)^3}{n^k+(n+2)^k}$ (xii) $\frac{\sqrt{n}}{n+1}$

2. Prove that the series

$\frac{1}{3}+\frac{1.4}{3.6}+\frac{1\cdot4.7}{3\cdot6.9}+\cdots$... is divergent but the series

$\left(\frac{1}{3}\right)^2+\left(\frac{1.4}{3.6}\right)^2+\left(\frac{1.4.7}{3.6.9}\right)^2+\cdots\ldots$ is convergent.

3. Use the inequality $e^x > x$ if $x > 0$ to show that the series Σe^{-n^2} converges.

4. Show that if $\sum a_n$ is convergent then $\sum a_n^2$, $\sum \frac{a_n}{1+a_n}$ and $\sum \frac{a_n}{1+n^2 a_n}$ are also convergent.

5.If $\sum a_n$ is a divergent series of positive terms, prove that $\sum \frac{a_n}{1+n^2 a_n}$ is convergent.

Unit IV

Kummer's test – Root test – Integral Test.

Chapter 4: Sections 4.1 - 4.3

4.1. Kummer's Test:

Theorem 1:(Kummer's test)

Let $\sum\ a_n$ be a given series of positive terms and $\sum \frac{1}{d_n}$ be a series of positive terms diverging to ∞. Then

(i) $\sum\ a_n$ converges if $\lim_{n\to\infty}\left(d_n\frac{a_n}{a_{n+1}} - d_{n+1}\right) > 0$ and

(ii) $\sum\ a_n$ diverges if $\lim_{n\to\infty}\left(d_n\frac{a_n}{a_{n+1}} - d_{n+1}\right) < 0$.

Proof:

(i) Let $\lim_{n\to\infty}\left(d_n\frac{a_n}{a_{n+1}} - d_{n+1}\right) = l > 0$.

We distinguish two cases.

Case (i) l is finite:

Then given $\varepsilon > 0$, there exists $m \in N$ such that

$l - \varepsilon < d_n\dfrac{a_n}{a_{n+1}} - d_{n+1} < l + \varepsilon$ for all $n \geq m$

$\therefore\ d_na_n - d_{n+1}a_{n+1} > (l-\varepsilon)a_{n+1}$ for all $n \geq m$.

Taking $\varepsilon = \frac{1}{2}l$, we get $d_na_n - d_{n+1}a_{n+1} > \frac{1}{2}la_{n+1}$ for all $n \geq m$.

Now, let $n \geq m$.

$$\therefore d_ma_m - d_{m+1}a_{m+1} > \frac{1}{2}la_{m+1}$$

$$d_{m+1}a_{m+1} - d_{m+2}a_{m+2} > \frac{1}{2}la_{m+2}$$

$d_{n-1}a_{n-1} - d_na_n > \frac{1}{2}la_n$.

Adding. we get

$$d_ma_m - d_na_n > \frac{1}{2}l(a_{m+1} + \cdots \ldots + a_n).$$

$\therefore\ d_na_m - d_na_n > \dfrac{1}{2}l(s_n - s_m)$ where $s_n = a_1 + a_2 + \cdots \ldots + a_n$

$\therefore\ d_ma_m > \dfrac{1}{2}l(s_n - S_m)$.

$\therefore s_n < \frac{2d_m a_m + l s_m}{l}$ which is independent of n.

$\therefore$ The sequence (s_n) of partial sums is bounded.

$\therefore a_n$ is convergent.

Case (ii) $l = \infty$.

Then given any real number $k > 0$ there exists a positive integer m.

such that $d_n\left(\frac{a_n}{a_{n+1}}\right) - d_{n+1} > k$ for all $n \geq m$.

$\therefore d_n a_n - d_{n+1}a_{n+1} > k a_{n+1}$ for all $n \geq m$.

Now, let $n \geq m$. Writing the above inequality for

$m, m+1, \ldots \ldots (n-1)$ and adding we get

$$d_m a_m - d_m a_n > k(a_{m+1} + \cdots .. + (a_n)$$
$$= k(s_n - s_m).$$
$$\therefore d_m a_m > k(s_m - s_m).$$
$$\therefore s_n < \frac{d_m a_m}{k} + s_m$$

$\therefore$ The sequence (s_n) is bounded and hence Σa_n is convergent.

(ii) $\lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) = 1 < 0$

Suppose l is finite.

Choose $\varepsilon > 0$ such that $l + \varepsilon < 0$.

Then there exists $m \in$ N such that

$$l - \varepsilon < d_n \frac{a_n}{a_{n+1}} - d_{n+1} < l + \varepsilon < 0 \text{ for all } n \geq m$$

$\therefore d_n a_n < d_{n+1} a_{n+1}$ for all $n \geq m$

Now, let $n \geq m$. Then $d_m a_m < d_{m+1} a_{m+1}$

$d_{m+1} a_{m+1} < d_{m+2} a_{m+2}$

$d_{n-1} a_{n-1} < d_n a_n$

$\therefore d_m a_m < d_n a_n$.

$\therefore a_n > \frac{d_m a_m}{d_n}$.

Also, by hypothesis $\sum \frac{1}{d_n}$ is divergent.

Hence $\sum_{n=1}^{\infty} \frac{d_m a_m}{d_n}$ is divergent.

$\therefore$ By comparison test $\sum a_n$ is divergent.

The proof is similar if $l = -\infty$.

Note 1:

The above test fails if $\lim_{n\to\infty} \left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) = 0$.

Note 2:

The divergence of $\Sigma(1/d_n)$ has not been used in the proof of (i).

Corollary 1 (D 'Alembert's ratio test)

Let $\sum a_n$ be a series of positive terms. Then Σa_n converges if.

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} > 1$ and diverges if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$.

Proof:

The series $1 + 1 + 1$ is divergent. We can put $d_n = 1$ in Kummer's Test.

Then $d_n \frac{u_n}{a_n+1} - d_{n+1} = \frac{a_n}{a_{n+1}} - 1$

$\therefore \Sigma a_n$ converges if $\lim_{n\to\infty} \left(\frac{a_n}{a_{n+1}} - 1\right) > 0$.

$\therefore \sum a_n$ converges if $\lim_{n\to x} \frac{a_n}{a_n+1} > 1$.

Similarly $\sum a_n$ diverges if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$.

Corollary 2: (Raabe's Test)

Let $\sum a_n$ be a scries of positive terms. Then $\sum a_n$ converges if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1$ and diverges if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1$.

Proof:

The series $\sum \frac{1}{n}$ is divergent.

$\therefore$ We can put $d_n = n$ in Kummer's test.

Then $d_n \frac{a_n}{a_{n+1}} - d_{n+1} = n\frac{a_n}{a_{n+1}} - (n+1)$

$= n\left(\frac{a_n}{a_{n+1}} - 1\right)$ 1.

$\therefore \Sigma a_n$ converges if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1$ and diverges if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1$.

Corollary 3. (De Morgan and Bertrand's test)

Let $\sum a_n$ be a series of positive terms.

Then Σa_n is convergent if $\lim_{n\to\infty} \log n \left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] > 1$ and is divergent if

$\lim_{n\to\infty} \log n \left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] < 1$.

Proof:

The series $\sum \frac{1}{n\log n}$ is divergent. (This is proved later.)

∴ We can put $d_n = n\log n$ in Kummer's test.

Then $d_n \frac{a_n}{a_{n+1}} - d_{n+1} = (n\log n)\frac{a_n}{a_{n+1}} - (n+1)\log(n+1)$

$= \log n \left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] + (n+1)\log n - (n+1)\log(n+1)$

$= \log n \left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] - (n+1)\log\left(\frac{n+1}{n}\right)$.

∴ $\lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right)$

$$= \lim_{n\to\infty} (\log n)\left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] - \lim_{n\to\infty} \log\left(1 + \frac{1}{n}\right)^{n+1}$$
$$= \lim_{n\to\infty} (\log n)\left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] - 1.$$

∴ The result follows by applying Kummer's test.

Note:

The following is a more general form of Kummer's test.

Let $\sum a_n$ be a given series of positive terms and $\sum \frac{1}{d_n}$ be a series of positive terms diverging to ∞.

Then (i) Σa_n converges if $\liminf\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) > 0$

and (ii) $\sum a_n$ diverges if $\limsup\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) < 0$.

Thus D' Alembert's ratio test, Raabe's test, DeMorgan and Bertrand's test can be put in the more general form by replacing "limit" by "lim inf" and " lim sup" as the case may be.

Theorem 2: (Gauss's Test)

Let Σa_n be a series of positive terms such that $\frac{a_n}{a_{n+1}} = 1 + \frac{\beta}{n} + \frac{r_n}{n^p}$ where $P > 1$and (r_n) is a bounded sequence. Then the series Σa_n converges if $\beta > 1$ and diverges if $\beta \leq 1$.

Proof:

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\beta}{n} + \frac{r_n}{n^p}$$

$\therefore n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{\beta}{n} + \frac{r_n}{n^p}\right) = \beta + \frac{r_n}{n^{p-1}}$.

Now, since $\rho > 1$, $\lim_{n\to\infty} \frac{1}{n^{n-1}} = 0$.

Also (r_n) is a bounded sequence.

Hence $\lim_{n\to\infty} \frac{r_n}{n^{n-1}} = 0$ (by solved problem 4 of 3.6).

$$\therefore \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \beta$$

$\therefore$ By Raabes's test $\sum a_n$ converges if $\beta > 1$ and $\sum a_n$ diverges if $\beta < 1$.

If $\beta = 1$, Raabes's test fails. In this case we apply Kummer's test by taking $d_n = n\log n$.

Now, $d_n \frac{a_n}{a_{n+1}} - d_{n+1}$

$$= n\log n\left(1 + \frac{1}{n} + \frac{r_n}{n^p}\right) - (n+1)\log(n+1)$$

$$= -(n+1)\log\left(1 + \frac{1}{n}\right) + \frac{r_n \log n}{n^{p-1}}$$

$$= -\log\left(1 + \frac{1}{n}\right)^{n+1} + \frac{r_n \log n}{n^{p-1}}$$

Now, by hypothesis (r_n) is a bounded sequence and by problem 9 of 1.7 $\left(\frac{r_n \log n}{n^{p-1}}\right) \to 0$

$\left(\frac{r_n \log n}{n^{p-1}}\right) \to 0$

$$\therefore \lim_{n\to\infty} \left(a_n \frac{a_n}{a_{n+1}} - a_{n+1}\right) = -\log e = -1 < 0.$$

$\therefore$ By Kummer's tesi $\sum a_n$ diverges.

Note:

Let (a_n) be any sequence (b_n) be a sequence of positive real numbers. We say that (a_n) is of the same order of magnitude as (b_n) if there exists a real number k

such that $|a_n| < kb_n$ for all n and in this case we write $a_n = O(b_n)$.

In particular if $\left(\frac{a_n}{b_n}\right)$ is a convergent sequence then $a_n = O(b_n)$.

For example if $a_n = \frac{1}{(n+1)(n+2)}$ then $a_n = O(1/n^2)$.

Now Gauss's test can be restated as follows.

Let $\sum a_n$ be a series of positive terms such that $\frac{a_n}{a_{n+1}} = 1 + \frac{\beta}{n} + O\left(\frac{1}{n^p}\right)$ where $p > 1$. Then $\sum a_n$ converges if $\beta > 1$ and diverges il $\beta \leq 1$.

Problem 1:

Test the convergence of the series $\frac{1}{3} + \frac{1.2}{3.5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \cdots$ ...

Solution:

Let $a_n = \frac{1.2 \cdot 3 \ldots\ldots n}{3.5 \cdot 7 \ldots\ldots (2n+1)}$.

$$\therefore \frac{a_n}{a_{n+1}} = \frac{2n+3}{n+1} = \frac{2+3/n}{1+1/n}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} - 2 > 1.$$

∴ By I)' Alembert's ratio test Σa_n is convergent.

Problem 2:

Test the convergence of $\Sigma \frac{n^n}{n!}$.

Solution:

Let $a_n = \frac{n^n}{n!}$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{(n+1)n^n}{(n+1)^{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n}.$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{e} < 1.$$

$$\therefore \sum a_n \text{ is divergent.}$$

Problem 3:

Test the convergence of the series $\sum \frac{2^n n!}{n^n}$.

Solution:

Let $a_n = \frac{2^n n!}{n^n}$.

$$\therefore \frac{a_n}{a_{n+1}} = \frac{(n+1)^{n+r}}{2(n+1)n^n} = \frac{1}{2}\left(1+\frac{1}{n}\right)^n.$$

$$\therefore \lim_{n\to\omega} \frac{a_n}{a_{n+1}} = \frac{e}{2} > 1.$$

∴ By ratio test the series converges.

Problem 4:

Test the convergence of the series $\sum \frac{3^n n!}{n^n}$.

Solution:

As in the above problem, we find that $\lim_{n\to=2} \frac{a_n}{a_n,1} = \frac{e}{3} = 1.$.

∴ By ratio test the series diverges.

Problem 5:

Test the convergence of the series $\sum \sqrt{\frac{n}{n+1}} x^n$ where x is any posilive real number.

Solution:

Since x is positive the given series is a series of positive terms.

$$\frac{a_n}{a_{n+1}} = \sqrt{\frac{n(n+2)}{(n+1)}}\left(\frac{1}{x}\right)$$

Now,
$$= \sqrt{\frac{(1+2/n)}{1+1/n}}\left(\frac{1}{x}\right).$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{x}.$$

$\therefore$ By ratio test $\sum a_n$ converges if $x < 1$ and diverges if $x > 1$.

If $x = 1$ the test fails.

When $x = 1, a_n = \sqrt{\frac{n}{n+1}} = \frac{1}{\sqrt{(1+1/n)}}$

$\therefore \lim_{n\to\infty} a_n = 1$

$\therefore$ The series diverges.

Problem 6:

Test the convergence of the series

$1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots ..$ where x is any positive real number.

Solution:

Since x is a positive real number, the given series is a series of positive terms.

Let $a_n = \frac{x^{2n-2}}{2n-2}, (n > 1)$.

$$\therefore \frac{a_n}{a_{n+1}} = \frac{2n}{2n-2}\left(\frac{1}{x^2}\right).$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{x^2}.$$

$\therefore$ By ratio test, the series converges if $x^2 < 1$ and diverges if $x^2 > 1$. $\therefore$ The series converges if $x < 1$ and diverges if $x > 1$.

If $x = 1$ the test fails.

When $x = 1, a_n = \frac{1}{2n-2}$.

By comparing with the series $\Sigma(1/n)$ we see that the series diverges.

Problem 7:

Test the convergence of the series $\sum \frac{n^2+1}{5^n}$.

Solution:

$\frac{a_n}{a_{n+1}} = \frac{5(n^2+1)}{(n+1)^2+1}$

$= \frac{5(n^2+1)}{n^2+2n+2}$

$= \frac{5\left(1+\frac{1}{n^2}\right)}{1+\frac{2}{n}+\frac{2}{n^2}}$

$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 5.$

$\therefore$ By ratio test the series converges.

Problem 8:

Test the convergence of the series

$\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\ldots\ldots\ldots\ldots$

Solution:

Let $a_n = \frac{1}{2^n}+\frac{1}{3^n}$

$= \frac{2^n+3^n}{2^n 3^n}.$

$\therefore \frac{a_n}{a_{n+1}} = \frac{6(2^n+3^n)}{2^{n+1}+3^{n+1}}.$

$= \frac{2[1+(2/3)^n]}{[1+(2/3)^{n+1}]}.$

$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 2.$

$\therefore$ By ratio test the given series converges.

Problem 9:

Test the convergence of the series $\sum \frac{x^n}{n}$.

Solution:

Let $a_n = \frac{x^n}{n}.$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{n+1}{n}\left(\frac{1}{x}\right).$$

$$= \left(1+\frac{1}{n}\right)\left(\frac{1}{x}\right).$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{x}.$$

$\therefore$ The series converges if $x < 1$ and diverges if $x > 1$.

If $x = 1$, the series becomes $\sum \frac{1}{n}$ which is divergent.

Problem 10:

Test the convergence of the series $\sum \frac{n^p}{n!}(p > 0)$.

Solution:

Let $a_n = \frac{n^p}{n!}$.

$\therefore \frac{a_n}{a_{n+1}} = \frac{n^p(n+1)}{(n+1)^p}$.

$$= \frac{n+1}{(1+1/n)^p}$$

$$\therefore \lim_{n\to x} \frac{a_n}{a_n+1} = \infty$$

$\therefore$ By ratio test $\sum\ a_n$ is convergent.

Problem 11:

Test the convergence of the series

$$\frac{1}{3}x + \frac{1}{3}\frac{2}{5}x^2 + \frac{1}{3}\frac{2}{5}\cdot\frac{3}{7}x^3 + \cdots \ldots \ldots$$

Solution:

Let $a_n = \frac{1\cdot2\cdot3\ldots n}{3\cdot5\cdot7\ldots(2n+1)}x^n$.

$$\therefore \frac{a_n}{a_{n+1}} =$$

$$= \frac{2n+3}{n+1}\left(\frac{1}{x}\right)$$

$$= \frac{2+3/n}{1+1/n}\left(\frac{1}{x}\right).$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{2}{x}.$$

$\therefore$ By ratio test the series converges if $\frac{2}{x} > 1$.

$\therefore$ The series converges if $x < 2$ and diverges if $x > 2$.

If $x = 2$, the ratio test fails.

In this case, $\frac{a_n}{a_{n+1}} = \frac{2n+3}{2n+2}$.

$\therefore \frac{a_n}{a_{n+1}} - 1 = \frac{1}{2n+2}$.

$\therefore n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{n}{2n+2} = \frac{1}{2+2/n}$.

$\therefore \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{1}{2}$.

$\therefore$ By Raabe's test the series diverges.

Problem 12:

Test the convergence of the hyper geometric series

$$1 + \frac{\alpha\beta}{r}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{r(r+1)2!}x^2 + \cdots \ldots$$

Solution:

Let $a_n = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{r(r+1)\ldots(r+n-1)n!}x^n$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{(r+n)(n+1)}{(\alpha+n)(\beta+n)}\left(\frac{1}{x}\right)$$

$$= \frac{\left(1+\frac{r}{n}\right)\left(1+\frac{1}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}\left(\frac{1}{x}\right)$$

$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{x}$.

$\therefore$ The series converges if $x < 1$ and diverges if $x > 1$.

When $x = 1$, the ratio test fails.

In this case we apply Gauss' test.

$$\frac{a_n}{a_{n+1}} = \frac{\left(1+\frac{r}{n}\right)\left(1+\frac{1}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}$$

$$= \left(1+\frac{r}{n}\right)\left(1+\frac{1}{n}\right)\left(1+\frac{\alpha}{n}\right)^{-1}\left(1+\frac{\beta}{n}\right)^{-1}$$

$$= \left(1+\frac{r}{n}\right)\left(1+\frac{1}{n}\right)\left[1-\frac{\alpha}{n}+O\left(\frac{1}{n^2}\right)\right]\left[1-\frac{\beta}{n}+O\left(\frac{1}{n^2}\right)\right]$$

$$= 1 + \frac{(r+1-\alpha-\beta)}{n} + O\left(\frac{1}{n^2}\right)$$

$\therefore$ By Gauss' test the series converges if $r > \alpha + \beta$ and diverges if $r \leq \alpha + \beta$. Hence the given series

(i) converges if $x < 1$

(ii) diverges if $x > 1$

(iii) converges if $x = 1$ and $r > a + \beta$

(iv) diverge if $x = 1$ and $r \leq (1 + \beta)$.

Problem 13:

Test for convergence of the series whose n fernll is giventy

$$a_n = \frac{1^2 \cdot 3^2 \cdot 5^2 \ldots (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \ldots (2n)^2}$$

Solution:

$$\frac{u_n}{a_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}$$

$$= \left(1 + \frac{1}{n}\right)^2 \left(1 + \frac{1}{2n}\right)^{-2}$$

$$= \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\left[1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right]$$

$$= 1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

By Gauss's test the given series is divergent.

Exercises:

Test the convergence of the following series.

(1) $\sum \frac{n}{2^n}$ (2) $\sum \frac{.5^n}{n^2+5}$ (3) $1 + \frac{1.3}{1.4} + \frac{1 \cdot 3.5}{1 \cdot 4 \cdot 7} +$ (4) $1 + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} +$

(5) $\sum \frac{x^n}{\sqrt{(2n+3)}}$ (6) $1 + a + \frac{a(a+1)}{2!} + \frac{(a+1)(a+2)}{3!} +$ (7) $\frac{1}{3}x + \frac{2!}{3 \cdot 5}x^2 + \frac{3!}{3 \cdot 5 \cdot 7}x^3 +$

(8) $1 + \frac{3}{7}x + \frac{3 \cdot 6}{7 \cdot 10}x^2 + \frac{3 \cdot 6 \cdot 9}{7 \cdot 10 \cdot 13}x^3 +$ (9) $\sum \frac{\sqrt{n}}{n+1}x^n$ (10) $\sum \frac{x^{2n+1}}{\sqrt{(2n+3)}}$

4.2. Root Test and Condensation Test:

Theorem 1: (Cauchy's root test)

Let $\sum a_n$ be a series of penitive terms. Then $\sum a_n$ is convergent if $\lim_{n \to \infty} a_n^{1/n} < 1$ and divergent if $\lim_{n \to \infty} a_n^{1/n} > 1$.

Proof:

Case (i) Let $\lim_{n \to \infty} a_n^{1/n} = l < 1$

Choose $\varepsilon > 0$ such that $l + \varepsilon < 1$.

Then there exists $m \in N$ such that $a_n^{1/n} < l + \varepsilon$ for all $n \geq m$.

$\therefore a_n < (l + \varepsilon)^n$ for all $n \geq m$.

Now, since $l + \varepsilon < 1, \dot{\Sigma}(l + \varepsilon)^n$ is convergent.

(by example 2 of 3.1)

$\therefore$ By comparison test Σa_n is convergent.

Case (ii) Let $\lim_{n\to\infty} a_n^{1/n} = l > 1$.

Choose $\varepsilon > 0$ such that $l - \varepsilon > 1$.

Then there exists $m \in N$ such that $a_n^{1/n} > l - \varepsilon$ for all $n \geq m$.

$\therefore a_n > (l - \varepsilon)^n$ for all $n \geq m$.

Now, since $l - \varepsilon > 1, \Sigma(l - s)^n$ is divergent (by example 2 of 3.1).

$\therefore$ By comaprison test, Σa_n is divergent.

Note:

The following is a more general form of Cauchy's root test.

Let $\sum a_n$ be a series of positive terms. Then Σa_n is convergent if $\limsup a_n^{l/n} < 1$ and divergent if $\limsup a_n^{1/n} > 1$.

Theorem 2: (Cauchy's condensation text)

Let $a_1 + a_2 + a_3 + \ldots\ldots + a_n + \cdots \ldots \ldots$ (1)be a series of positive terms and whose terms are monotonic decreasing. Then this series converges or diverges according as the series

$ga_g + g^2 a_g^2 + \cdots \ldots \ldots \ldots + g^n a_z^n + \qquad \ldots\ldots\ldots\ldots\ldots\ldots$ (2)

converges or diverges where g is any positive integer > 1.

Proof:

Let $s_n = a_1 + a_2 + \quad + a_n$ and

$t_s = ga_g + g^2 a_g^2 + \cdots \ldots \ldots \ldots + g^n a_g^n$.

Then $s_g^n = \left(a_1 + a_2 + \cdots \ldots \ldots + a_g\right) + \left(a_{g+1} + a_{g+2} + \cdots \ldots \ldots + a_g^2\right) +$

$\ldots\ldots\ldots\ldots + \left(a_{g+1}^{n-1} + a_{g+2}^{n-1} + \cdots \ldots \ldots + a_g^n\right)$

$\leq ga_1 + (g^2 - g)a_g + \cdots \ldots\ldots + (g^n - g^{n-1})a_g^{n-1}$.

(since the terms of the series are monotonic decreasing).

$= ga_1 + g(g-1)a_g + g^2(g-1)a_g^2 + \cdots \ldots \ldots + g^{n-1}(g-1)a_g^{n-1}$

$= ga_1 + (g-1)\left(ga_g + g^2a_g^2 + \dots\dots\dots\ +g^{n-1}a_g^{n-1}\right.$

$= ga_1 + (g-1)t_{n-1}.$

$\therefore s_g^n \leq ga_1 + (g-1)t_{n-1}.$

$\therefore$ If the series (2) converges, then (1) converges.

Now, $s_g^n \geq ga_g + (g^2-g)a_g^2 + \ \ +(g^n - g^{n-1})a_g^n$

$$= ga_g + \frac{g-1}{g}\left(g^2a_z^2 + \cdots \dots \dots + g^n a_{gn}\right)$$

$= ga_g + \frac{g-1}{g}\left(t_n - ga_g\right) = a_g + \frac{g-1}{g}t_n.$

$\therefore$ If the series (2) diverges, then (1) diverges.

Problem 1:

Test the convergence of $\sum \frac{1}{(\log n)^n}$

Solution:

Let $a_n = \frac{1}{(\log n)^n}$

$\therefore \sqrt[n]{a_n} = \frac{1}{\log n}.$

$\therefore \lim \sqrt[n]{a_0} = 0 < 1.$

$\therefore$ By Cauchy's root test $\sum \frac{1}{(\log n)^n}$ converges.

Problem 2:

Test the convergence of $\sum \left(1+\frac{1}{n}\right)^{-n}$

Solution:

Let $a_n = \left(1+\frac{1}{n}\right)^{-n}$

$$\therefore \sqrt[n]{a_n} = \left(1+\frac{1}{n}\right)^{-n}$$

$\therefore \lim_{n\to-\infty} \sqrt[n]{a_n} = \frac{1}{e}$ (refer problem 3 of 1.7)

< 1

$\therefore$ By Cauchy's root test the series converges.

Problem 3:

Prove that the series $\sum\ e^{-\sqrt{n}}x^n$ converges if $0 < x < 1$ and diverges if $x > 1$.

Solution:

Let $a_a = e^{-\sqrt{n}}x^n.$

$\therefore a_n^{1/n} = e^{-1/\sqrt{n}} x.$

$\therefore \lim_{n \to -} a_n^{\nu n} - x.$

$\therefore$ By Cauchy's root test the given series converges if $0 < x < 1$ and diverges if $x > 1$.

Problem 4:

Test the convergence of $\sum \frac{n^3+a}{2^n+a}$.

Solution:

Let $a_n = \frac{n^3+a}{2^n+a}$ and $b_n = \frac{n^3}{2^n}$

$$\therefore \frac{a_n}{b_n} = \left(\frac{n^3 + a}{2^n + a}\right)\left(\frac{2^n}{n^3}\right)$$

$$= \left(\frac{n^3 + a}{n^3}\right)\left(\frac{2^n}{2^n + a}\right)$$

$$= \left(1 + \frac{a}{n^3}\right)\left(\frac{1}{1 + (a/2^n)}\right)$$

$\lim_{n \to \infty} \frac{a_n}{b_n} = 1.$

$\therefore$ By comparison test, the given series is convergent or divergent according as $\sum \frac{n^3}{2^n}$ is convergent or divergent.

Now, $\sqrt[n]{b_n} = \left(\frac{n^3}{2^n}\right)^{1/n} = \frac{n^{3/n}}{2}.$

Also $\lim n^{3/n} = 1.$

$\therefore \lim_{n \to -\infty} \sqrt[n]{b_n} = \frac{1}{2}.$

$\therefore \Sigma b_n$ is convergent.

$\therefore \Sigma a_n$ is convergent.

Problem 5:

Test the convergence of $\sum \frac{1}{n\log n}$.

Solution:

By Cauchy's condensation test, $\sum \frac{1}{n\log n}$ converges or diverges with the series.

$$\sum \frac{2^n}{2^n \log 2^n} = \sum \frac{1}{n\log 2} = \frac{1}{\log 2} \sum \frac{1}{n}$$

Now, the series $\sum \frac{1}{n}$ diverges.

∴ The given series diverges.

Problem 6:

Test the convergence of the series $\sum \frac{1}{n(\log n)^p}$.

Solution:

The given series converges or diverges with the series

$$\sum \frac{2^n}{2^n(\log 2^n)^p} = \sum \frac{1}{(\log 2)^p n^p}$$
$$= \frac{1}{(\log 2)^p} \sum \frac{1}{n^p}$$

The series $\sum \frac{1}{n}$ converges if $p > 1$ and diverges if $p \leq 1$.

∴ The given series converges if $p > 1$ and diverges if $p \leq 1$.

Problem 7:

Test the convergence of the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$ ...

Solution:

We have $a_n^{1/n} = \begin{cases} \left(\frac{1}{3^{n/2}}\right)^{1/n} & \text{If } n \text{ is even} \\ \left(\frac{1}{2^{(n+1)/2}}\right)^{1/n} & \text{If } n \text{ is odd} \end{cases}$

$$a_n^{1/n} = \begin{cases} \frac{1}{\sqrt{3}} & \text{If } n \text{ is even} \\ \frac{1}{2^{1/2(1+\frac{1}{n})}} & \text{If } n \text{ is odd} \end{cases}$$

Now, the sequence $\frac{1}{2^{1/2(1+\frac{1}{n})}}$ converges to $\frac{1}{\sqrt{2}}$ as $n \to -\infty$.

$\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{2}}$ are the only limit points of the given sequences $\limsup a_n^{\frac{l}{n}} = \frac{1}{\sqrt{2}} < 1$.

By Cauchy's root test the given series is convergent.

4.3. Integral Test:

Theorem 1:(Cauchy's integral test)

Let f be a non-negative monotonic decreasing integrable function defined on $[1,\infty)$. Let $l_n = \int_1^n f(x)dx$. Then the series $\Sigma f(n)$ converges iff the sequence (I_n) converges. Further the sum of the series lies between $l = \lim_{n\to\infty} I_n$ and $I + f(1)$.

Proof:

Let $f(n) = a_n$. Since f is monotonic decreasing $f(n-1) \geq f(x) \geq f(n)$ where $n-1 \leq x \leq n$.

$\therefore a_{n-1} \geq f(x) \geq a_n$

$$\therefore \int_{n-1}^{n} a_{n-1}dx \geq \int_{n-1}^{n} f(x)dx \geq \int_{n-1}^{n} a_n dx$$

$a_{n-1} \geq \int_{n-1}^{n} f(x)dx \geq a_n$ ………… (1)

Replacing n by $2,3, \dots \dots \dots, n$ in (1) and adding we obtain

$$a_1 + a_2 + \cdots \dots + a_{n-1} \geq \int_1^n f(x)dx \geq a_2 + a_3 + \cdots .. + a_n$$

$\therefore s_n - a_n \geq I_n \geq s_n - a_1$ where $s_n = a_1 + a_2 + \cdots \dots \dots + a_n$

$\therefore a_1 \geq s_n - I_n \geq a_n$

Now, since f is non-negative, $f(n) = a_n \geq 0$.

$\therefore a_1 \geq s_n - I_n \geq 0$.

Now, let $s_n - I_n = A_n$.

$\therefore a_1 \geq A_n \geq 0$. …………(2)

$\therefore (A_n)$ is a bounded sequence.

Also $A_{n+1} - A_n = s_{n+1} - s_n - I_{n+1} + I_n$

$$= a_{n+1} - \int_n^{n+1} f(x)dx$$

$$\leq a_{n+1} - \int_n^{n+1} a_{n+1}dx$$

$$\leq 0$$

$\therefore A_{n+1} \leq A_n$.

$\therefore A_n$ is a bounded monotonic decreasing sequence.

$\therefore \lim A_n = \lim(s_n - I_n)$ exists.

$\therefore \lim_{n\to-\infty} s_n$ exists iff $\lim I_n$ exists and $\lim A_n = s - I$ ……….(3)

where s is the sum of the series and $I = \lim_{n\to-\infty} I_n$.

$\therefore$ The series $\sum f(n)$ converges iff the sequence (I_n) converges.

In this case from (2) $a_1 \geq \lim A_n \geq 0$.

$\therefore a_1 \geq s - I \geq 0$ (by (3))

$\therefore I + a_1 \geq s \geq I$.

$\therefore I + f(1) \geq s \geq I$.

Problem 1:

Show that $\lim_{n\to-\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\ldots.+\frac{1}{n}-\log n\right)$ exists and lies helwecn 0 a and 1. (This limit is known as Euler's constant and denoted by v).

Solution:

Consider the function $j(x) = 1/x$ defined on $[1,\infty)$. Clearly $f(x)$ is non-negative and monotonic decreasing.

$I_n = \int_1^n \frac{1}{x}dx = \log n$.

Let $f(n) = a_n = 1/n$.

$\therefore s_n - I_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots\ldots.+\frac{1}{n}-\log n$.

Now by Cauchy's integral test $s_n - I_n$ converges and its limit lies belween 0 and $\boldsymbol{a}_1$.

But $a_1 = f(1) = 1$

$\therefore \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots\ldots\ldots+\frac{1}{n}-\log n\right)$ exists and lies between 0 and 1.

Problem 2:

Discuss the convergence of the series $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^a}$ where $\boldsymbol{x} \geq 0$.

Solution:

Let $a_n = \frac{1}{n(\log n)^a}\alpha \geq 0, n \geq 2$.

Consider the function $f(x) = \frac{1}{x(\log x)^a}$ so that $f(n) = a_n$.

Clearly $f(x)$ is non-negative and monotoaic decreasing on $[2,\infty)$.

Case (i) Let $\alpha \neq 1$.

$$\therefore I_n = \int_2^n \frac{dx}{x(\log x)^\alpha}$$

$$= \left[\frac{1}{1-\alpha}(\log x)^{1-\alpha}\right]_2^n$$

$$= \frac{(\log n)^{1-\alpha}}{1-\alpha} - \frac{(\log 2)^{1-\alpha}}{1-\alpha},$$

$\therefore (I_n)$ converges if $\alpha > 1$ and diverges if $\alpha < 1$.

Hence by Cauchy's integral test, the given series converges if $\alpha > 1$ and diverges if $\alpha < 1$.

Case (ii) Let $\alpha = 1$.

$\therefore I_n = [\log(\log x)]_2^n$

$= [\log(\log n) - \log(\log 2)] \to \infty$ as $n \to \infty$.

$\therefore (I_n)$ diverges and bence the given series diverges.

Problem 3:

Using the integral test discuss the convergence of the series $\sum ne^{-n^2}$

Solution:

Let $a_n = ne^{-n^2}$.

Consider the function $f(x) = xe^{-x^2}$ so that $f(n) = a_n$. Clearly $f(x)$ is non-negative and monotonic decreasing on $[1, \infty)$.

Also $I_n = \int_1^n xe^{-x^2} dx$.

$= \frac{1}{2}\left(e^{-1} - e^{-n^2}\right)$.

$\therefore I_n \to \frac{1}{2}e^{-1}$ as $n \to \infty$.

$\therefore$ By Cauchy's integral test, the given series is convergent and its suml lics between $\frac{1}{2}e^{-1}$ and $\frac{3}{2}e^{-1}$.

Exercises.

1. Show that the series $\sum \frac{1}{n}$ converges if $p > 1$ and diverges if $p \leq 1$ and in case of convergence the sum lies between $\frac{1}{p-1}$ and $\frac{p}{p-1}$.
2. Discuss the convergence of the following series using Cauchy's integral test.

   (i) $\sum_1^\infty \frac{1}{n^2+1}$ (ii) $\sum_1^\infty \frac{1}{n(\log n)^2}$

   (iii) $\sum_3^\infty \frac{1}{n\log n(\log\log n)^2}$ (iv) $\sum_1^\infty \frac{1}{(n+1)^2}$

   (v) $\sum_1^\infty \frac{n^4}{2n^5+3}$ (vi) $\sum_1^\infty \frac{1}{n(n+1)}$

   (vii) $\sum_1^\infty \frac{1}{\sqrt{(n^2-1)}}$.

Unit V

Series of Arbitrary terms: Alternative series – Absolute convergence – Tests for convergence of series of arbitrary terms.

Chapter 5: Sections 5.1 - 5.3

5.Series of Arbitrary Terms:

So far we have been dealing with series of positive terms. We now consider series in which the terms are not necessarily positive.

5.1 Alternating Series:

Definition. A series whose terms are alternatively positive and-negative is called an alternating series.

Thus an altering series is of the form

$a_1 - a_2 + a_3 - a_4 + \cdots \ldots \ldots \ldots \ldots\ldots\ldots = \Sigma(-1)^{n+1}a_n$ where $a_n > 0$ for all n.

For example

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \ldots = \sum\ (-1)^{n+1}\left(\frac{1}{n}\right)$ is an alternating series.

(ii) $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots .. = \sum\ (-1)^{n+1}\left(\frac{n+1}{n}\right)$ is an alternating series.

We now prove a test for convergence of an alternating series.

Theorem 1: (Leibnitz's test)

Let $\Sigma(-1)^{n+1}a_n$ be an alternating series whose terms a_n satisfy the following conditions (i) (a_n) is a monotonic decreasing sequence.

(ii) $\lim_{n\to\infty} a_n = 0$.

Then the given alternating series converges.

Proof:

Let (s_n) denote the sequence of partial sums of the given series.

Then $s_{2n} = a_1 - a_2 + a_3 - a_1 + \ \ + a_{2n-1} - a_{2n}$

$s_{2n+2} = s_{2n} + a_{2n+1} - a_{2n+2}$

$\therefore s_{2n+2} - s_{2n} = (a_{2n+1} - a_{2n+2}) \geq 0$ (by (i)).

$\therefore s_{2n+2} \geq s_{2n}$.

$\therefore (s_{2n})$ is a monotonic increasing sequence.

Also $s_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots \ldots . -(a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1$ by (i).

$\therefore (s_{2_n})$ is bounded above.

$\therefore (s_{2n})$ is a convergent sequence

Let $(s_{2n}) \to s$.

Now, $s_{2n+1} = s_{2n} + a_{2n+1}$.

$\therefore \lim_{n\to-} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} a_{2n+1}$

$= s + 0 = s.(by(\text{ii}))$

$\therefore (s_{2n+1}) \to s$

Thus the subsequences (s_{2n}) and (s_{2n+1}) converge to the same limits.

$\therefore (s_n) \to s$ (by sec 2.3 theorem 2).

$\therefore$ The given series converges.

Note:

In the above theorem if $\lim_{n-\infty} a_n = a \neq 0$, then $\lim_{n-\infty} s_{2n} =$ s and $\lim_{n-\infty} s_{2n+1} = s + a$. Hence the sequence (s_n) cannot converge. Further (s_n) is $_a$ bounded sequence. Hence (s_n) oscillates.

$\therefore$ The given series oscillates.

Problem 1:

Show that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \dots \dots$ converge.

Solution:

The given series is $\Sigma(-1)^{n+1} a_n$ where $a_n = 1/n$

Clearly $a_n > a_{n+1}$, for all n and hence (a_n) is monotonic decreasing.

Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

$\therefore$ By Leibnitz's tesi the given series converges.

Problem 2:

Show that the series $\Sigma \frac{(-1)^{n+1}}{\log(n+1)}$ converges .

Solution:

Let $a_n = \frac{1}{\log(n+1)}$.

Clearly $(a_n) \to 0$ as $n \to \infty$.

Also $\frac{1}{\log n} > \frac{1}{\log(n+1)}$ for all $n \geq 2$.

$\therefore$ By Leibnitz's test the given series converges.

Problem 3:

Show that the series $\Sigma(-1)^{n+1}\frac{n}{3n-2}$ oscillates.

Solution:

Let $a_n = \frac{n}{3n-2}$.

Clearly $a_n > a_{n+1}$ for all n.

Also $\lim_{n\to\infty}\frac{\text{n}}{3n-2} = \frac{1}{3}$.

∴ The given series oscillates.

Problem 4:

Show that the following series converges

$\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3\_4)$+………

Solution:

Let $a_n = \frac{1+2+3+\cdots\ldots\ldots+n}{(\text{n}+1)^3}$

$$= \frac{n(n+1)}{2(\text{n}+1)^3}$$

$$= \frac{n}{2(\text{n}+1)^2}$$

Clearly $a_n > a_{n+1}$, for all n.

Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{2(\text{n}+1)^2}$

$$= \lim_{n\to\infty}\frac{1}{2\text{n}(1+1/\text{n})^2} = 0$$

∴ By Leibnitz's test the given series converges.

Exercises:

(1) $\sum\frac{(-1)^n(1+n^2)}{1+n^3}$

(2) $\sum(-1)^{-\left(1+\frac{1}{n}\right)}$

(3) $1 - \frac{1}{3}\left(1+\frac{1}{2}\right) + \frac{1}{5}\left(1+\frac{1}{2}+\frac{1}{3}\right) - \frac{1}{8}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{2}\right) + \cdots \ldots \ldots ..$

(4) $1 - \left(\frac{1}{2^2}+\frac{1}{3^2}\right) + \left(\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}\right) + \cdots \ldots \ldots ..$

(5) $\sum\frac{(-1)^{n-1}}{\sqrt{n}}$

(6) $\sum \frac{(-1)^{-2}\log(n+1)}{(n+1)^{-2}}$

(7) $\sum \frac{(-1)^n n}{2n-1}$

(8) $\sum \frac{(-1)^{n-1} n}{5^n}$.

(9) $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} -$

(10) $\sum (-1)^n \sin\left(\frac{1}{n}\right)$.

5.2. Absolute Convergence:

Definition:

A series Σa_n is said to be absolutely convergert if the series $\sum |a_n|$ is convergent. .

Examples.

1. The series $\sum \frac{(-1)^n}{n^2}$ is absolutely convergent, for, $\sum \left|\frac{(-1)^n}{n^2}\right| = \sum \frac{1}{n^2}$ which is convergent.
2. The series $\sum \frac{(-1)^n}{n}$ is not absolutely convergent for, $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ is divergent.

   However, the given series is convergent (by problem 1 of 5.1).

Note:

If Σa_n is a convergert sories of positive temms them Σa_n is absolutely convergent.

Theorem 1:

Any absolutely convergent series is convergent.

Proof:

Let Σa_n be absolutely convergent .

$\therefore \dot{\Sigma} |a_n|$ is convergent.

Let $s_n = a_1 + a_2 + \cdots \ldots \ldots .. + a_n$ and $t_n = |a_1| + |a_2| + \cdots \ldots \ldots \ldots + |a_n|$

By hypothesis (l_n) is convergent and hence is a Cauchy sequence.

Hence given $\varepsilon > 0$, there exists $n_1 \varepsilon N$ such that

$|t_n - t_m| < \varepsilon$ for all $n, m \geq n_1$ ....(1)

Now let $m > n$.

Then $|s_n - s_m| = |a_{n+1} + a_{n+2} + \cdots \ldots \ldots \ldots \ldots + a_m|$

$\leq |a_{n+1}| + |a_{n+2}| + \cdots \ldots \ldots \ldots\ldots + |a_m|$
$= |t_n - t_m|$
$< \varepsilon$ for all $n, m \geq n_1$(by (1)).

$\therefore$ (s_n) is a Cauchy sequence in **R** and hence is convergent..

$\therefore$ Σa_n is a convergent series.

Note 1:

The converse of the above theorem is not true. For example, the series $\sum (-1)^n \frac{1}{n}$ is convergent. However $\sum \frac{1}{n}$ is divergent so that the series is not absolutely convergent.

Note 2:

Since $\Sigma|a_n|$ is a series of positive terms, the tests developed in chapter 4 for series of positive terms can be used to test the absolute convergence of a given series.

Definition:

A series. $\sum a_n$ is said to be conditionally convergent if it is convergent but not absolutely convergent.

Example:

The series $\sum \frac{(-1)^n}{n}$ is conditionally convergent.

Theorem 2:

In an absolutely convergent series, the series formed by its positive terms alone is convergent and the series formed by its negative terms alone is convergent and conversely.

Proof:

Let $\sum a_n$ be the given absolutely convergent series.

We define $p_n = \begin{cases} a_n \text{ if } a_n > 0 \\ 0 \quad \text{if } a_n \leq 0 \end{cases}$ and

$q_n = \begin{cases} 0 \text{ if } a_n \geq 0 \\ -a_n \text{ if } a_n < 0 \end{cases}$

(i,e) p_n is a positive term of the given series and q_π is the modulus of a negative term

$\therefore$ Σp_n is the series formed with the positive terms of the given series and Σq_n is the series formed with the moduli of the negative terms of the given series.

Clearly $p_n \leq |a_n|$ and $q_n \leq |a_n|$ for all n.

Since the given series is absolutely convergent, $\Sigma|a_n|$ is a convergent series of positive terms.

Hence by comparison test Σp_n and Σq_n are convergent.

Conversely Σp_n and Σq_n converge to p and q respectively. We claim that $\sum a_n$ is absolutely convergent.

We have $|a_n| = p_n + q_n$

$\therefore \Sigma|a_n| = \Sigma(p_n + q_n)$
$\qquad = \Sigma p_n + \Sigma q_n$
$\qquad = p + q$

$\therefore \Sigma a_n$ is absolutely convergent.

Theorem 3:

If Σa_n is an absolutely convergent scries and (b_n) is a bounded sequence, then the series $\sum a_n b_n$ is an absolutely convergent series.

Proof:

Since (b_n) is a bounded sequence, there exists a real number $k > 0$

such that $|b_n| \leq k$ for all n.

$\therefore |a_n b_n| = |a_n||b_n| \leq k|a_n|$ for all n.

Since $\sum a_n$ is absolutely convergent $\Sigma|a_n|$ is convergent.

$\therefore \Sigma k|a_n|$ is convergent

$\therefore$ By comparison test $\Sigma|a_n b_n|$ is convergent.

$\therefore \Sigma a_n b_n$ is absolutely convergent.

Problem 1:

Test for convergence of the series $\sum \frac{(-1)^n}{n^p}$.

Solution:

Case (i) Let $p > 1$.

Then $\sum \left|\frac{(-1)^n}{n^p}\right| = \sum \frac{1}{n^p}$ is convergent.

$\therefore$ The given series is absolutely convergent and hence convergent.

Case (ii) Let $0 < p \leq 1$.

Then $\left(\frac{1}{n^p}\right)$ is a monotonic decreasing sequence converging to 0 .

$\therefore$ By Leibnitz's test the given series converges.

absolute convergence

In this case the convergence is not absolute since $\sum \frac{1}{n^0}$ diverges

when $0 < p \leq 1$.

Case (iii) Let $p = 0$. Then the series reduces to $-1 + 1 - 1 +$ which oscillates finitely.

Case (iv) Let $p < 0$. Then the sequence $\left(\frac{1}{n^p}\right)$ is unbounded. Hence the given scrics oscillates infinitely.

Problem 2:

Show that the series $\Sigma(-1)^n\left[\sqrt{(n^2+1)}-n\right]$ is conditionality convergent.

Solution:

Let $a_n=\sqrt{(n^2+1)}-n=\frac{1}{\sqrt{(n^2+1)}+n}$

Clearly (a_n) is a monotonic decreasing sequence converging to 0.

∴ By Leibnitz's test the given series converges.

Now we prove that $\Sigma\left|(-1)^n\left(\sqrt{(n^2+1)}-n\right)\right|$ is divergent.

$\left|(-1)^n\left(\sqrt{(n_2+1)}-n\right)\right|=a_n=\frac{1}{\sqrt{(n_2+1)}+n}$.

Let $b_n=1/n$.

$\therefore \frac{a_n}{b_n}=\frac{n}{\sqrt{(n^2+1)}+n}=\frac{1}{\sqrt{\left(1+\frac{1}{n^2}\right)}+1}$.

$\therefore \lim_{n\to\infty}\frac{a_n}{b_n}=\frac{1}{2}$.

∴ By comparison test $\sum a_n$ is divergent.

∴ The given series is not absolutely convergent.

∴ The given series is conditionally convergent.

series of arbitrary terms

Problem 3:

Show that the series $\sum \frac{x^{n-1}}{(n-1)!}$ converges absolutely for all values of x.

Solution:

Let $a_n=\frac{x^{n-1}}{(n-1)!}$.

$\therefore \left|\frac{a_n}{a_{n+1}}\right|=\frac{n}{|x|}$

$\therefore \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\infty$ for all $x\neq 0$.

∴ By ratio test the series $\sum\left|\frac{x^{n-1}}{(n-1)!}\right|$ is convergent for all $x\neq 0$ and the convergence is trivial for $x=0$.

∴ The series converges absolutely for all x.

Problem 4:

Test the convergence of $\sum \frac{(-1)^n \sin n\alpha}{n^3}$

Solution:

We have $\left|\frac{(-1)^n \sin n\alpha}{n^3}\right| \leq \frac{1}{n^3}$ (since $|\sin \theta| \leq 1$).

∴ By comparison test the series is absolutely convergent.

Exercises.

1. Discuss the convergence of the following series.

   (a) $\sum \frac{a+(-1)^n}{n^2}, a \in \mathbf{R}$.

   (b) $\sum \frac{(-1)^n x^n}{\log(n+1)}$

   (c) $\sum(-1)^n \left\{\frac{1}{n^2} + \frac{1}{(n+1)^2}\right\}$

   (d) $\sum \frac{(-1)^n (n+2)}{2^n+5}$

   (e) $\sum \frac{(-1)^n \cos n\alpha}{n\sqrt{n}}$

   (f) $\left(\frac{1}{2}\right)^2 - \left(\frac{1.3}{2.4}\right)^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 -$

   (g) $\sum \frac{(-1)^n x^n}{1+n\alpha}$

2. Show that in a conditionally convergent series the series formed by its positive terms alone is divergent and the series formed by its negative terms also is divergent.

5.3.Tests for Convergence of Series of Arbitrary Terms:

Some tests for establishing the convergence of series of arbitrary terms are given in this section.

Theorem 1:

Let (a_n) be a bounded sequence and (b_n) be a monotonic decreasing bounded sequence. Then the series $\Sigma a_n(b_n - b_{n+1})$ is absolutely convergent.

Proof:

Since (a_n) and (b_n) are bounded sequences there exists a real number $k > 0$ such that $|a_n| \leq k$ and $|b_n| \leq k$ for all n.

Let s_n denote the partial sum of the series $\Sigma|a_n(b_n - b_{n+1})|$.

$$\therefore s_n = \sum_{r=1}^{n} |a_r(b_r - b_{r+1})|$$

$$= \sum_{r=1}^{n} |a_r|(b_r - b_{r+1}) \text{ (since } b_r > b_{r+1} \text{ for all } r)$$

$$\leq k \sum_{r-1}^{n} (b_r - b_{r+1})$$

$$= k(b_1 - b_{n+1})$$

$$\leq k(|b_1| + |b_{n+1}|)$$

$$\leq k(k+k) = 2k^2.$$

$\therefore (s_n)$ is a bounded sequence.

$\therefore \sum \ |a_n(b_n - b_{n+1})|$ is convergent.

$\therefore \sum \ a_n(b_n - b_{n+1})$ is absolutely convergent.

Theorem 2: (Dirichlet's test)

Let $\sum \ a_n$ be a series whose sequence of partial sums (s_n) is bounded Let (b_n) be a monotonic decreasing sequence converging to 0 . Then the series $\Sigma a_n b_n$ converges.

Proof:

Let t_n denote the partial sum of the series $\sum \ a_n b_n$.

$$\therefore t_n = \sum_{r=1}^{n} a_r b_r$$

$$= s_1 b_1 + \sum_{r=2}^{n} (s_r - s_{r-i}) b_r \text{ (since } s_r - s_{r-1} = a_r)$$

$= \sum_{r=2}^{n} \ s_r(b_r - b_{r+1}) + s_n b_n$ .... (1)

Since (s_n) is bounded and (b_n) is a monotonic decreasing bounded n-1 sequence

$\sum_{r=1}^{n-1} \ s_r(b_r - b_{r+1})$ is a convergent sequence (by theorem 1)

Also since (s_n) is bounded and $(b_n) \rightarrow 0, (s_n b_n) \rightarrow 0.$

(by problem 4 of 1.6).

$\therefore$ From (1) it follows that (t_n) is convergent.

$\therefore \sum \ a_n b_n$ is convergent.

Note:

Leibnitz's test for alternating series proved in 5.1 is a particular case of Dirichlet's test. For, consider the alternating series $\Sigma(-1)^n a_n$ where (a_n) is a monotonic decreasing sequence converging to zero. The sequence of partial sums of $\Sigma(-1)^n$ is obviously a bounded sequence.

Hence by Dirichlet's test $\Sigma(-1)^n a_n$ converges.

Theorem 3: (Abel's test)

Let $\sum\ a_n$ be a convergent series. Let (b_n) be bounded monotonic sequence. Then $\sum\ a_n b_n$ is convergent

Proof:

Since (b_n) is a bounded monotonic sequence; $(b_n) \to b$ (say)

Let $c_n = \begin{cases} b - b_n \text{ if } (b_n) \text{ is monotonic increasing} \\ b_n - b \text{ if } (b_n) \text{ is monotonic decreasing} \end{cases}$

$a_n c_n == \begin{cases} a_n b - a_n b_n \text{ if } (b_n) \text{ is monotonic increasing} \\ a_n b_n - a_n b \text{ if } (b_n) \text{ is monotonic decreasing} \end{cases}$

$a_n b_n = \begin{cases} a_n b - a_n c_n \text{ if } (b_n) \text{ is monotonic increasing} \\ \text{b } a_n + a_n c_n \text{ if } (b_n) \text{ is monotonic decreasing} \end{cases}$. ………….(1)

Clearly (c_n) is a monotonic decreasing sequence converging to 0 .

Also since $\sum a_n$ is a convergent series its sequence of partial sums is bounded.

∴ By Dirichlet's test $\sum a_n c_n$ is convergent.

Also $\sum \dot{a}_n$ is convergent.

∴ $\Sigma b a_n$ is convergent.

∴ By(1), $\sum a_n b_n$ is convergent.

Problem 1:

Show that convergence of $\sum a_n$ implies the convergence of $\sum \frac{a_n}{n}$.

Solution:

Let $\Sigma a_n be$ convergen.

The sequence $\left(\frac{1}{n}\right)$ is a bounded monotonic sequence.

Hence by Abel's test $\sum \frac{a_n}{n}$ is convergent.

Problem 2:

Show that the series $\sum \frac{\sin n\theta}{n}$ converges for all values of θ and $\sum \frac{\cos n\theta}{n}$ converges if θ is not a multipic of 2π.

Solution:

Consider the series $\sum \frac{\sin n\theta}{n}$.

Let $a_n = \sin n\theta$ and $b_n = 1/n$.

Clearly (b_n) is a monotonic decreasing sequence converging to 0 .

Now, $s_n = \sin\theta + \sin 2\theta + \cdots \ldots + \sin n\theta$

$$= \frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\left[2\sin\theta\sin\frac{\theta}{2} + \cdots \ldots + 2\sin n\theta\sin\frac{\theta}{2}\right]$$

$$= \frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\left[\left(\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right) + \cdots\cdots + \left(\cos\left(\frac{2n-1}{2}\right)\theta - \cos\frac{2n+1}{2}\theta\right)\right]$$

$$= \frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\left[\cos\frac{\theta}{2} - \cos\left(\frac{2n+1}{2}\right)\theta\right]$$

$$\therefore |s_n| = \left|\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right|\left|\cos\frac{\theta}{2} - \cos\left(\frac{2n+1}{2}\right)\theta\right|$$

$$= \frac{1}{2}\left|\operatorname{cosec}\frac{\theta}{2}\right|\left[\left|\cos\frac{0}{2}\right| + \left|\cos\left(\frac{2n+1}{2}\right)\theta\right|\right]$$

$$\leq \frac{1}{2}\left|\operatorname{cosec}\frac{\theta}{2}\right| \times 2 = \left|\operatorname{cosec}\frac{\theta}{2}\right|$$

$\therefore |s_n| \leq \left|\operatorname{cosec}\frac{\theta}{2}\right|$.

$\therefore (s_n)$ is a bounded sequence when θ is not a multiple of 2π

$\therefore$ By Dirichlet's test $\sum a_n b_n = \sum \frac{\sin n\theta}{n}$ converges when θ is not a multiple of 2π.

When θ is a multiple of 2π, the series $\sum \frac{\sin n\theta}{n}$ reduces to $0 + 0 + 0 +$ which trivially converges to 0 .

$\therefore \sum \frac{\sin n\theta}{n}$ converges for all values of 0 .

Now, we consider the series $\sum \frac{\cos n\theta}{n}$.

$s_n = \cos\theta + \cos 2\theta + \cdots \ldots \ldots + \cos n\theta$

$= \frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\left[\sin\frac{2n+1}{2}\theta - \sin\frac{\theta}{2}\right]$.

$\therefore |s_n| \leq \left|\operatorname{cosec}\frac{\theta}{2}\right|$.

$\therefore (s_n)$ is a bounded sequence when 0 is not a multiple of 2π.

$\therefore$ By Dirichlet's test $\sum \frac{\cos n\theta}{n}$ converges when θ is not nultiple of 2π.

When 0. is a multiple of 2π, the series reduces $1 + \frac{1}{2} + \frac{1}{3} +$ which diverges.

$\therefore$ The series $\sum \frac{\cos n\theta}{n}$ converges except when θ is a mullipl of 2π.

Problem 3:

Prove that $\sum_{n=2}^{\infty}\left(\frac{\sin n}{\log n}\right)$ is convergent.

Solution:

Let $a_n = \sin n$ and $b_n = 1/\log n$.

Clearly (b_n) is a monotonic decreasing sequence converging to 0 .

$s_n = \sin 2 + \sin 3 + \cdots \ldots \ldots + \sin(n+1)$

$= \frac{1}{2}\operatorname{cosec}\frac{1}{2}\left[\cos\left(\frac{3}{2}\right) - \cos\left(\frac{2n+3}{2}\right)\right]$ (as in problem 2)

$\therefore |s_n| \leq \operatorname{cosec}\left(\frac{1}{2}\right)$

$\therefore$ (s_n) is a bounded sequence.

By Dirichlet's test $\sum_{n=2}^{\infty}\left(\frac{\sin n}{\log n}\right)$ converges.

Problem 4:

Discuss the convergence of the series $\sum\left(1+\frac{1}{2}+\cdots \ldots + \frac{1}{n}\right)\frac{\sin n\theta}{\mathrm{n}}$.

Solution:

Let $b_n = \left(\frac{1}{n}\left(1+\frac{1}{2}+\cdots \ldots + \frac{1}{n}\right)\right)$

And $a_n = \sin n\,\theta$

As in problem 1, the partial sum s_n of the series $\sum \sin n\theta$ is bounded except when θ is a multiple of 2π.

Now since $\frac{1}{n}$ is a monotonic decreasing sequence $\frac{1}{n}\left(1+\frac{1}{2}+\cdots \ldots + \frac{1}{n}\right)$ is also a monotonic decreasing sequence (refer problem 1 of 1.3)

Also by Cauchy's first limit theorem

$\left(\frac{1}{n}\left(1+\frac{1}{2}+\cdots \ldots + \frac{1}{n}\right)\right) \to 0.$

$\therefore$ By Dirichlet's test, the given series converges except when 0 is a multiple of 2π.

When θ is a multiple of 2π, the series reduces to $0 + 0 + \cdots$.. which converges to zero.

$\therefore$ The givenseries converges for all values of 0 .

Exercises:

1. Show that the convergence of $\Sigma a_n \Rightarrow$ the convergence of

   (i) $\sum \frac{a_n}{\log n}$

   (ii) $\sum \frac{n+1}{n} a_n$

(iii) $\Sigma n^{1/n} a_n$

(iv) $\sum \left(1+\frac{1}{n}\right)^n a_n$

2. Show that the series $\sum \left(1+\frac{1}{2}+\cdots..+\frac{1}{n}\right)\frac{\cos n\theta}{n}$ converges except when is multiple of 2π.

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